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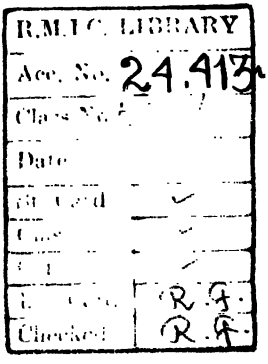
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CONTENTS

MATHEMATICS

		PAGE
1.	On the Diffraction of Light by a Transparent Wedge—By Prof. S. K. Banerji, D.Sc.	1
2.	On a geometrical treatment of the scattering of light by a perfectly reflecting cone—By Abanibhusan Datta, M.A.	9
3.	A Note on the Deformation of Surfaces—By Prof. B. M. Sen, M.Sc., B.A. (Cant.)	15
4.	On the Stability of two co-axial rectilinear vortices of Compressible fluid—By Bibhutibhusan Datta, M.Sc.	21
5.	On Some Properties of Natural Numbers—By Haripada Datta, M.A.	31
6.	A Note on Whittaker's Formula for the Solution of Algebraic or Transcendental Equations—By Bholanath Pal, M.A.	41
7.	Potent divisors of the characteristic matrix of a minimum simple square ante-slope—By Nripendranath Ghosh, M.A.	45
8.	Liquid motion inside certain rotating curvilinear rectangles —By Nripendranath Sen, M.Sc.	51
9.	On the Horpolhode—By Satyendranath Basu, M.Sc. ...	65
10.	On a Generalization of Neumann's Expansion in a Series of Bessel Functions—By Abanibhusan Datta, M.A.	67
11.	A Note on the Theory of the Magnetometer—By L. Srivastava, M.Sc.	79
12.	On the motion of an Elongated Spheroid in Viscous Fluid Media—By Bholanath Pal, M.A.	87
13.	Cullis's Matrices and Determinoids—By Rai Bahadur A. C. Bose, M.A.	95
14.	Note on Spherical Waves of Finite Amplitude—By Prof. S. K. Banerji, D.Sc.	139

PAGE

15. On the Theory of Continued Fraction—By Haripada Datta, M.A. 147
16. Evaluation of the product matrix in a commutantal product of simple matrices having given nullities—By Prof. C. E. Cullis, M.A., Ph.D. 157
17. A note on the construction by points of a trinodal quartic —By Girindralal Mukherjee, M.A., B.L. ... 203
18. On Liquid Motion inside Rotating Arcs of Three and Four Confocal Parabolas—By Nalinimohan Basu, M.Sc. ... 209
19. Investigations on Ripple Motion—By J. C. Kamesvara Rav, M.Sc. 225
20. Note on the product of any number of Legendre functions of different degrees—By Prof. S. K. Banerji, D.Sc. ... 231

PHYSICS

1. Ionisation in the Solar Chromosphere—By Dr. Meghnad Saha, D.Sc. 1
2. On the Application of Electron-Chemistry to Problems of Radiation and Astrophysics—By Dr. Meghnad Saha, D.Sc. 17
3. A Report on the Theories of the Specific Heat of Solids—By Sasibhusan Mali, M.Sc. 25

BOTANY

Commentationes Mycologicæ.

1. *Pseudoperonospora cubensis* (B. & C.) Rosten, on *Trichosanthes dioica* Roxb.—By S. N. Bal, M.Sc. 1
2. *Cercospora Personata* (B. & C.), Ellis, on leaves of *Arachis hypogæa*, Linn.—By S. N. Bal, M.Sc. 4
3. *Rhinocladium corticolum*, Mass., on the bark of *Mangifera Indica*, Linn.—By S. N. Bal and K. G. Banerjee ... 7

MATHEMATICS

On the Diffraction of Light by a Transparent Wedge*

BY

SUDHANSUKUMAR BANERJI

1. Although the problem of the diffraction of light by a perfectly reflecting and perfectly absorbing wedge has been very successfully treated by Sommerfeld and other writers including Jackson[1] Macdonald,[2] Reiche,[3] Wiegrefe,[4] Bromwich[5] and Whipple[6] by the use of a certain type of contour integral, the problem of the diffraction of light by a transparent wedge, when the velocity of radiation in the body differs by a finite amount from that in the surrounding medium does not appear to have received so much attention. There are in fact not many cases in which the influence of the material properties of the obstacle has been taken into account in the mathematical treatment of a diffraction problem although the necessity of doing this has been very clearly indicated by some of the experimental results. It was therefore considered that the treatment of the problem of diffraction of light by a dielectric wedge given in this paper may not probably be altogether devoid of interest. After this paper was read, Prof. Carslaw has published an important paper on the diffraction of light by a perfectly conducting wedge of any angle in the *Proceedings of the London Mathematical Society*, December, 1919. The present paper had to be re-written in view of the results obtained by Prof. Carslaw in this paper.

2. Let the edge of the wedge be chosen as the axis of z and let r, θ, z be the cylindrical co-ordinates of a point so that the faces of the wedge are given by $\theta=0$ and $\theta=\beta$ and that the space occupied by it is that between $\theta=\beta$ and $\theta=2\pi$. Taking the case in which the electric force

* Reprinted from the *Bulletin of the Calcutta Mathematical Society*, Vol. X, No. 4.

[1] Jackson, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1 (1904), p. 393.

[2] Macdonald, *Electric Waves*, Appendix D (1902); *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 14 (1915), p. 410.

[3] Reiche, *Ann. d. Phys.*, Bd. 37 (1912), p. 131.

[4] Wiegrefe, *Ann. d. Phys.*, (4), Bd. 39 (1912), p. 449.

[5] Bromwich, *Proc. Lond. Math. Soc.*, Vol. 14 (1915), p. 450.

[6] Whipple, *Proc. Lond. Math. Soc.*, Vol. 16 (1917), p. 94.

in the incident wave is parallel to the edge of the wedge, it is easy to see that the electric force in the total disturbance outside the wedge and that in the disturbance which enters into the body of the wedge must both be parallel to the edge of the wedge. Let Z_0 denote the electric force in the incident wave, then

$$Z_0 = e^{ik_1[r\cos(\theta-\theta')+V_1 t]},$$

which represents a set of plane waves coming from the direction θ'. Prof. Carslaw[1] has transformed this expression into the integral

$$\frac{1}{2\pi}\int e^{ik_1[r\cos(\alpha-\theta)+V_1 t]} \frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha,$$

over the path (A) in the α—plane (see the figure in his paper). He has also obtained the following expression which is a finite and continuous solution of the wave equation and of period $2n\pi$, n being any ptiveosi integer,

$$Z=\frac{1}{2n\pi}\int e^{ik_1[r\cos(\alpha-\theta)+V_1 t]} \frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha,$$

the integral being taken over the same path (A) in the α—plane. This integral has the property that as $r\longrightarrow\infty$, $Z\longrightarrow Z_0$, when $|\theta-\theta'|<\pi$, and $Z\longrightarrow 0$ when $\pi<|\theta-\theta'|<2n\pi$. A solution of period 2β is similarly obtained in the form

$$\frac{1}{2\beta}\int e^{ik_1[r\cos(\alpha-\theta)+V_1 t]} \frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta}-e^{i\pi\theta'/\beta}}\,d\alpha,$$

over the same path (A) in the α—plane. These periodic solutions of the wave equation have been used by Prof. Carslaw in the discussion he has recently given of the problem of the diffraction of waves by a perfectly conducting wedge without making any use of Riemann's surfaces.

A more general solution of the wave equation of period 2β can be obtained by starting from the integral

$$\frac{1}{2n\pi}\int f[r\cos(\alpha-\theta)+V_1 t] \frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha,$$

[1] See his paper on "Multiform Solutions" in the *Proc. Lond. Math. Soc.*, Ser. 1, Vol. 30 (1898), p. 121.

over the path (A) in the α—plane, where $f[r\cos(\theta-\theta')+V_1 t]$ is an arbitrary continuous solution of the wave equation

$$\frac{\partial^2 Z}{\partial r^2}+\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}-\frac{1}{V_1^2}\frac{\partial^2 Z}{\partial t^2}=0.$$

By making n tend to infinity and by proceeding as in §3 of Prof. Carslaw's paper already quoted, we can easily reduce the integral to the form

$$\frac{1}{2i\pi}\int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta+V_1 t)\,\frac{2\zeta}{\zeta^2-(\theta-\theta')^2}\,d\zeta,$$

where $2\pi>\gamma>\pi$, $0>\gamma'>-\pi$ and by adding to this the solutions of the same type which correspond to the directions

$$\theta'\pm 2\beta,\ \theta'\pm 4\beta,\ldots.$$

we obtain the following solution of period 2β.

$$\frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta+V_1 t)\,\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')}\,d\zeta.$$

Similarly by adding the solutions which correspond to waves in the directions

$$-\theta'\pm 2\beta,\ -\theta'\pm 4\beta,\ldots,$$

we obtain the solution

$$\frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta+V_1 t)\,\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\,d\zeta,$$

where $$2\pi>\gamma>\pi,\ 0>\gamma'>-\pi.$$

3. If now Z_1 denotes the electric force in the disturbance outside the wedge and Z_2 that in the disturbance inside the wedge, then since at the boundary between two media, the tangential electric force must be continuous, the following condition must be satisfied on the two faces of the wedge :

$$Z_1 = Z_2, \text{ when } \theta = 0 \text{ and } \theta = \beta.$$

that is. $$Z_1 - Z_2 = 0, \text{ when } \theta = 0 \text{ and } \theta = \beta.$$

We can therefore assume the following expressions for Z_1 and Z_2 :

$$Z_1 = \frac{Ai}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta + V_1 t)} \left[\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} - \frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta,$$

$$+ \frac{i}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} f(r\cos\zeta + V_1 t) \left[\frac{C\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{D\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta.$$

$$Z_2 = \frac{i}{2\beta} \int_{\infty i+\gamma}^{\infty i+\gamma'} F(r\cos\zeta + V_2 t) \left[\frac{E\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{F\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta} - \cos\frac{\pi}{\beta}(\theta+\theta')} \right] d\zeta.$$

V_1, V_2 denoting the velocities of propagation in the two media.

If now we suppose that the two arbitrary functions F and f satisfy the equation

$$F(r\cos\zeta+V_2t)=f(r\cos\zeta+V_1t).$$

for all values of $r\cos\zeta$ and t, then the condition

$$Z_1-Z_2=0, \text{ when } \theta=0 \text{ and } \theta=\beta.$$

is satisfied if $C=F$ and $D=E$. The alternative solution $C=-D$ and $E=-F$ is obviously inadmissible as we are then led to the condition that Z_1 and Z_2 separately vanish on the two faces of the wedge $\theta=0$ and $\theta=\beta$. It is also easy to see that if we choose the two arbitrary functions to have the forms

$$f(r\cos\zeta+V_1t)=e^{ik_1(r\cos\zeta+V_1t)},$$

and

$$F(r\cos\zeta+V_2t)=e^{ik_2(r\cos\zeta+V_2t)}$$

$$+\frac{k_1-k_2}{k_2}\left[ik_2(r\cos\zeta+V_2t)\right]e^{ik_2(r\cos\zeta+V_2t)}$$

$$+\frac{(k_1-k_2)^2}{2!\,k_2^2}\left[ik_2(r\cos\zeta+V_2t)\right]^2e^{ik_2(r\cos\zeta+V_2t)}$$

$$+\ldots.$$

then the two functions are very nearly identical for all values of $r\cos\zeta$ and t, provided that

$$k_1V_1=k_2V_2$$

and k_1, k_2 differ only by a small quantity. When k_1 and k_2 differ by a finite quantity, the form of the function F can be obtained by the method of successive approximations.

We thus obtain the following expressions for Z_1 and Z_2:

$$Z_1 = \frac{Ai}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1 t)}\left[\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')}\right.$$

$$\left.-\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\right]d\zeta,$$

$$+\frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1 t)}\left[\frac{C\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')}\right.$$

$$\left.+\frac{D\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\right]d\zeta,$$

$$Z_2 = \frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} F(r\cos\zeta+V_2 t)\left[\frac{D\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')}\right.$$

$$\left.+\frac{C\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\right]d\zeta.$$

4. If (R_1, Θ_1) denote the components of magnetic force in the directions r and θ in the disturbance outside the wedge and (R_2, Θ_2) those in the directions r, θ in the disturbance inside the wedge, we have

$$-i\mu_1 R_1 = \frac{1}{r}\frac{\partial Z_1}{\partial\theta}, \quad i\mu_1\Theta_1 = \frac{\partial Z_1}{\partial r},$$

$$-i\mu_2 R_2 = \frac{1}{r}\frac{\partial Z_2}{\partial\theta}, \quad i\mu_2\Theta_2 = \frac{\partial Z_2}{\partial r},$$

μ_1, μ_2 being the magnetic permeabilities of the two media.

Since at the boundary of two media, the tangential components of the magnetic force must be continuous, we have

$$R_1 - R_2 = 0, \text{ when } \theta = 0 \text{ and } \theta = \beta.$$

This gives

$$\frac{2A}{\mu_1} + \frac{C-D}{\mu_1} + \frac{C-D}{\mu_2} = 0,$$

that is, $$C - D = -2A\mu_2 \div (\mu_1 + \mu_2).$$

The condition that at the boundary between two media the normal magnetic induction must be continuous, that is to say,

$$\mu_1 \Theta_1 - \mu_2 \Theta_2 = 0, \text{ when } \theta = 0 \text{ and } \theta = \beta,$$

is easily seen to be automatically satisfied.

Also as r tends to infinity

$$Z_1 \longrightarrow e^{ik_1[r\cos(\theta - \theta') + V_1 t]}.$$

It will thus appear that the above expressions for Z_1 and Z_2 satisfy all the conditions of the problem and it will also be seen that we have used one more constant than are actully required for the solution of the problem. We can therefore impose one relation between C and D consistent with the conditions of the problem. We can therefore assume

$$C + D = A - 1.$$

Thus we get

$$C = \tfrac{1}{2}(A-1) - \frac{A\mu_2}{\mu_1 + \mu_2},$$

$$D = \tfrac{1}{2}(A-1) + \frac{A\mu_2}{\mu_1 + \mu_2}.$$

We thus finally obtain the following expressions for Z_1 and Z_2 :

$$Z_1 = \frac{Ai}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1 t)}\left[\frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')} - \frac{\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\right]d\zeta,$$

$$+\frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} e^{ik_1(r\cos\zeta+V_1 t)}\left[\frac{\left\{\frac{1}{2}\left(A-1-\frac{A\mu_2}{\mu_1+\mu_2}\right)\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{\left\{\frac{1}{2}\left(A-1+\frac{A\mu_2}{\mu_1+\mu_2}\right)\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\right]d\zeta,$$

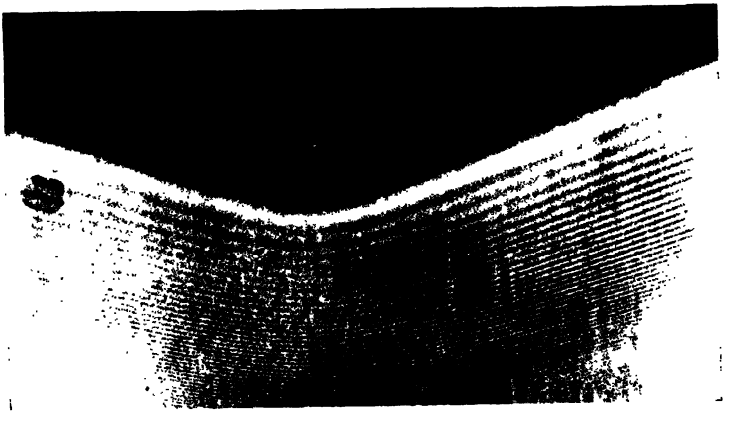
$$Z_2 = \frac{i}{2\beta}\int_{\infty i+\gamma}^{\infty i+\gamma'} F(r\cos\zeta+V_2 t)\left[\frac{\left\{\frac{1}{2}(A-1)+\frac{A\mu_2}{\mu_1+\mu_2}\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta-\theta')} + \frac{\left\{\frac{1}{2}(A-1)-\frac{A\mu_2}{\mu_1+\mu_2}\right\}\sin\frac{\pi\zeta}{\beta}}{\cos\frac{\pi\zeta}{\beta}-\cos\frac{\pi}{\beta}(\theta+\theta')}\right]d\zeta.$$

For the absolute determination of A we have to equate the energy contained in the primary wave with the energy contained in the disturbance functions Z_1 and Z_2. This will give a linear equation for the determination of A.

When the wedge is perfectly conducting, we have $Z_2=0$ at all points inside the wedge This gives $C=0$, $D=0$ and $A=1$. The expression for Z_1 then agrees with the well-known solution for a perfectly conducting wedge.

In the particular case when the dielectric properties of the two media differ very little from each other, the expression for Z_2 becomes considerably simplified, as we have already seen, for we can then neglect all powers of k_1-k_2 beyond the first.

The modification to be made in the treatment given above for the associated problem when the magnetic force in the incident wave is parallel to the edge of the wedge can be easily obtained.



Illustrating the scattering of light by a perfectly reflecting cone.

On a geometrical treatment of the scattering of light by a perfectly reflecting cone

[With a Plate]

BY

ABANIBHUSAN DATTA

§ 1. The present paper is an attempt to explain the form of the diffraction fringes observed when a perfectly reflecting cone is exposed to a point source of light by a geometrical method which though not very accurate from the point of view of the wave theory is still found to be capable of explaining the phenomena. If the diffraction fringes are photographed on a plate placed behind the cone, it is found that the figures are approximately a series of hyperbolas having their asymptotes parallel to the two lines which are the projection of the cone on the plate. This will be evident from the plate annexed to this paper which has been kindly lent by Prof. C. V. Raman to the present writer.

A rigorous mathematical theory of the problem of the scattering of sound waves by a cone has been given by Prof. Carslaw.[1] But the solution obtained by him does not appear to be easily amenable to numerical calculation.

I wish to express my obligation to Prof. S. K. Banerji for the many valuable suggestions, I have obtained from him regarding this problem.

§ 2. If A be the source of light, a ray falling on the surface of the cone at the point B, wi l be reflected in the plane containing the incident ray and the normal and making an equal angle with this

[1] *Math. Annalen*, Vol. 75 (1914), p. 143.

normal as the incident ray according to the law of geometrical optics. Whether a point on the reflected ray coincides with a maximum or minimum intensity will be determined by the path difference (AB + BP − AP). If we put this quantity $=n\lambda$, where λ is the wave length and n is any integer and if we determine the locus of P, the locus will correspond to a curve of maximum or minimum intensity. This locus can be obtained by a simple geometrical construction. If we conceive a set of confocal ellipsoids of revolution constructed having the points A and P as foci and AP as the axis of revolution, one and only one of this set will touch the cone at the point B. Conversely, it follows that if we construct an ellipsoid of revolution about an axis AP having A as one of its foci and touching the cone at the point B, then P will be its other focus. Now if k be the major axis of such an ellipsoid of revolution, and e the eccentricity, the expression for the path difference can be written in the form $k-ke=n\lambda$. It is therefore evident that if we construct all the possible ellipsoids of revolution which can touch the cone, having one point A which is the source of light as a fixed focus and a variable second focus P, the locus of the variable focus P will be a curve either of maximum or minimum intensity.

§ 3. Let the vertex of the cone be taken as the origin and the plane determined by the given focus and the axis of the cone as the xy-plane. Then the equation of the cone can be written in the form

$$fx^2+y^2+z^2=0 \qquad \ldots \quad (1)$$

Let the co-ordinates of the given focus be $(a, \beta, 0)$ and those of the variable focus be (a, b, c). Hence the equation of the ellipsoid is

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}+\sqrt{(x-a)^2+(y-\beta)^2+z^2}=k.$$

On rationalization, this is equivalent to

$$k^2\{(a-a)^2-k^2\}+4y^2\{b-\beta)^2-k^2\}+4z^2(c^2-k^2)+8yz(b-\beta)c$$

$$+8zx(a-a)c+8xy(a-a)(b-\beta)+4\{x(a-a)+y(b-\beta)+zc\}$$

$$\{k^2+(a^2+\beta^2)-(a^2+b^2+c^2)\}+8k^2(ax+\beta y)+[\{k^2-(a^2+\beta^2)$$

$$-(a^2+b^2+c^2)\}^2-4k^2(a^2+\beta^2)]=0. \qquad \ldots \quad (2)$$

Let the discriminant of (2) be Δ, and A, B, C its leading first minors. Then

$$\Delta=-64k^6\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2\}^2 \quad \ldots \quad (3)$$

$$A=16k^4\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2\}\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2-4a\alpha\}.$$

For brevity let us put $N=k^2-(a-\alpha)^2-(b-\beta)^2-c^2$

Hence
$$\left.\begin{aligned} A&=16k^4N(N-4a\alpha)\\ B&=16k^4N(N-4b\beta)\\ C&=16k^4N^2. \end{aligned}\right\} \quad \ldots \quad (4)$$

The second minors (a, b, c, d, l, m, n on the left-hand-side having their usual meanings as constituents of Δ) are given by

$$\left.\begin{aligned} al-l^2&=-4k^2[N(N-4b\beta)-4\beta^2c^2]\\ bd-m^2&=-4k^2[N(N-4a\alpha)-4\alpha^2c^2]\\ cd-n^2&=-4k^2[N(N-4a\alpha-4b\beta)-4(a\beta-b\alpha)^2] \end{aligned}\right\} \quad \ldots \quad (5)$$

§. 4 The condition that (1) should touch (2) is that the discriminant of the biquadratic

$$\lambda^4\Delta+\lambda^3\Theta+\lambda^2\Phi+\lambda\Theta'+\Delta'=0 \quad \ldots \quad (6)$$

should vanish, where, on calculation

$$\left.\begin{aligned} \Theta&=16k^4N[(f+2)N-4f\alpha-4b\beta].\\ \Phi&=-4k^2[2(f+1)N^2-4(2f\alpha+fb\beta+b\beta)N-4c^2(f\alpha^2+\beta^2)-4f(a\beta-b\alpha)^2],\\ \Theta'&=f[\{k^2+(\alpha^2+\beta^2)-(a^2+b^2+c^2)\}^2-4k^2(\alpha^2+\beta^2)], \end{aligned}\right\} \quad (7)$$

$$\Delta'=0.$$

The discriminant of (6) on simplification reduces to

$$\Theta'^2\{\Theta^2\Phi^2-4\Theta^3\Theta'+18\Delta\Theta\Phi\Theta'-27\Delta^2\Theta'^2-4\Delta\Phi^3\} \quad \ldots \quad (8)$$

But since $\Theta \neq 0$ (that being the condition that the ellipsoid passes through the vertex of the cone which is a particular case), the condition reduces to vanishing of D, the expression within the bracket in (8).

§ 5. Now $$k-ke=n\lambda$$

$$k=n\lambda+ke=n\lambda+\sqrt{(a-\alpha)^2+(b-\beta)^2+c^2} \quad \ldots (9)$$

Hence the locus of (a, b, c), the variable focus, is obtained by eliminating k between (9) and D. Neglecting the common factor $4^6k^{12}\{k^2-(a-\alpha)^2-(b-\beta)^2-c^2\}^2$ in D, we get the locus to be a surface of the sixth degree. On such a surface therefore, the light is either minimum or maximum.

§ 6. Section of the locus by the plane $y=$constant is obtained by putting $b=d$ (say) in D. From (9), putting $b=d$,

$$k^2-(a-\alpha)^2-(d-\beta)^2-c^2=n^2\lambda^2+2n\lambda\sqrt{(a-\alpha)^2+(d-\beta)^2+c^2}$$

$$=M \text{ (say)}. \quad \ldots (10)$$

Eliminating k by means of (10), leaving out the common factor we get

$$\Delta=-64M^2,\ \Theta=16M\{M(f+2)-4fa\alpha-4d\beta\}.$$

$$\Phi=-4\{M^2(2f+1)-4M(2fa\alpha+fd\beta+d\beta)-4c^2(fa^2+\beta^2)-4f(a\beta-d\alpha)^2\},$$

$$\Theta'=f[M^2-(4d\beta+4a\alpha)M-4(a\beta-d\alpha)^2-4c^2(a^2+\beta^2)].$$

Now, when the focus A is at infinity (*i.e.*, when the rays of light are parallel), a, β, and N are all infinite and we can put for them $l\infty$, $m\infty$ and $2n\lambda.\infty$ respectively. Then leaving out the common factor in D and dividing by $(\infty)^6$, we get the locus in the form

$$\mathfrak{D}_2=\Theta_1^2\Phi_1^2-4\Phi_1^3-8n\lambda\Theta_1^3\Theta'_1+36n\lambda\Theta_1\Phi_1\Theta'_1-108n^2\lambda^2\Theta_1^2=0, \quad (11)$$

where

$$\Theta_1 = 2n\lambda(f+2) - 4fl\,x - 4dm,$$

$$\Phi_1 = 4n^2\lambda^2\,(2f+1) - 8n\lambda(2fl\,x + fdm + dm) - 4x^2(l^2+m^2) - 4f(lm - dl)^2,$$

$$\Theta'_1 = f[4n^2\lambda^2 - 8n\lambda(dm + l\,x) - 4(lm - dl)^2 - 4x^2].$$

By giving different integral values to n, we get the different curves corresponding to maximum or minimum intensity.

§ 7. Since $n\lambda$ is in general a very small quantity, the last three terms in (11) can be neglected in comparison with the first two and the section of the locus by the plane y=constant is therefore approximately given by

$$\Theta_1^2\Phi_1^2 - 4\Phi_1^3 = 0,$$

or

$$\Phi_1^2(\Theta_1^2 - 4\Phi_1) = 0$$

The second degree terms in $(\Theta_1^2 - 4\Phi_1)$ are

$$16(l^2+m^2)(x^2+z^2).$$

Hence $\Theta_1^2 - 4\Phi_1 = 0$ represents a set of hyperbolas, the directions of the asymptotes of which are $x^2+z^2=0$ which are the same as those given by the section of the cone by the plane.

It appears that the curves represented by $\Phi_1^2 = 0$ do not belong to the particular phenomena discussed in this paper.

§ 8. Now if the light is supposed to come from a direction making an angle of 60° with the axis and if the cone is supposed to have a semivertical angle of 45°, then $f=-1$. $l=\frac{1}{2}$, $m=\sqrt{\frac{3}{2}}$. Let $\lambda=$ ·0006mm, $n=1$, $d=10{,}000\ k\lambda$ where $k=\sqrt{3}$. For these substitutions.

$$\Theta_1 = 2x - 35\cdot9988$$

$$\Phi_1 = 3x^2 - 2z^2 - 35\cdot9952x + 107\cdot9999.$$

Hence $\Theta_1^2 - 4\Phi_1 \equiv x^2 - z^2 - 108 = 0$ ... (12)

and the direction of the asymptotes of this are given by

$$x^2 - z^2 = 0.$$

It is to be noted that by giving to n, the values 2, 3, 4 etc. in succession, we shall get a series of hyperbolas which correspond to the curves of maximum or minimum intensity.

A Note on the Deformation of Surfaces

BY

BHUPATIMOHAN SEN

It is usually stated in the standard works on Differential Geometry that the problem of deformation depends on a partial differential equation of the second order of the Monge—Ampere type. Given the Cartesian co-ordinates of a surface in terms of two parameters (u, v) it has been proved that each of the co-ordinates satisfies the differential equation

$$(x_{11}-x_1\Gamma-x_2\Delta)(x_{22}-x_1\Gamma''-x_2\Delta'')-(x_{12}-x_1\Gamma'-x_2\Delta')^2$$

$$=k^2\{V^2-(Gx_1^2-2Fx_1x_2+Ex_2^2)\}. \qquad \ldots \quad (1)$$

[Forsyth, Diff. Geom., p. 363]

When one of the co-ordinates, say x, has been determined by this equation, the others can be determined by quadrature from the equations

$$\left.\begin{aligned} x_1^2+y_1^2+z_1^2&=E,\\ x_2^2+y_2^2+z_2^2&=G,\\ x_1x_2+y_1y_2+z_1z_2&=F. \end{aligned}\right\} \qquad \ldots \quad (2)$$

But there is nothing to shew that the values of y and z thus obtained after the substitution of the value of x will satisfy the Monge-Ampere equation which must be satisfied by the Cartesian co-ordinates of all surfaces having the same E, F and G.

The orthodox solution of the problem of deformation is therefore incomplete. In fact it is of doubtful value. The equation of deformation is a necessary but not a sufficient condition as will now be shown.

The real solution will be obtained by eliminating two of the three independent variables in the simultaneous equations

$$x_1^2+y_1^2+z_1^2=E,$$
$$x_1x_2+y_1y_2+z_1z_2=F,$$
$$x_2^2+y_2^2+z_2^2=G.$$

But the straightforward elimination is impracticable.

Let us introduce the new independent variables D, D', D'' given by

$$\left.\begin{aligned} D&=Xx_{11}+Yy_{11}+Zz_{11}\\ D'&=Xx_{12}+Yy_{12}+Zz_{12}\\ D''&=Xx_{22}+Yy_{22}+Zz_{22}\end{aligned}\right\}, \tag{3}$$

X, Y, Z being $\frac{1}{V}\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}$, $\frac{1}{V}\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}$ and $\frac{1}{V}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$, where

$$V=+\sqrt{EG-F^2}.$$

We now look upon the whole problem from the stand-point of differential equations.

From equations (2) we obtain by differentiation[1]

$$\left.\begin{aligned} \Sigma x_1x_{11}&=\tfrac{1}{2}\frac{\partial E}{\partial u}, \quad \Sigma x_2x_{11}=\frac{\partial F}{\partial u}-\tfrac{1}{2}\frac{\partial E}{\partial v},\\ \Sigma x_1x_{12}&=\tfrac{1}{2}\frac{\partial E}{\partial v}, \quad \Sigma x_2x_{12}=\tfrac{1}{2}\frac{\partial G}{\partial u},\\ \Sigma x_1x_{22}&=\frac{\partial F}{\partial v}-\tfrac{1}{2}\frac{\partial G}{\partial u}, \quad \Sigma x_2x_{22}=\tfrac{1}{2}\frac{\partial G}{\partial v}\end{aligned}\right\} \tag{4}$$

Solving for x_{11}, x_{12}, x_{22} between equations (3) and (4),

$$\left.\begin{aligned} x_{11}&=\Gamma x_1+\Delta x_2+DX,\\ x_{12}&=\Gamma' x_1+\Delta' x_2+D'X\\ x_{22}&=\Gamma'' x_1+\Delta'' x_2+D''X.\end{aligned}\right\} \tag{5}$$

[1] This is the standard method for the deduction of the Gaussian and the Codazzi relations. It has been repeated in outline for emphasising the analytical stand-point.

Using now the conditions of integrability of the type

$$\frac{\partial}{\partial v}(x_{11}) = \frac{\partial}{\partial u}(x_{12}),$$

we get the three fundamental equations first obtained by Gauss and Codazzi:

$$\frac{DD''-D'^2}{V^2} = \frac{1}{2V}\left\{\frac{\partial}{\partial v}\left[\frac{F}{EV}\frac{\partial E}{\partial v} - \frac{1}{V}\frac{\partial G}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{2}{V}\frac{\partial F}{\partial u} - \frac{1}{V}\frac{\partial E}{\partial v} - \frac{F}{EV}\frac{\partial E}{\partial u}\right]\right\}, \quad \ldots \quad (6)$$

$$\left.\begin{aligned} -\frac{\partial}{\partial v}\left(\frac{D'}{V}\right) - \frac{\partial}{\partial u}\left(\frac{D''}{V}\right) + \Delta''\frac{D}{V} - 2\Delta'\frac{D'}{V} + \Delta\frac{D''}{V} &= 0, \\ \frac{\partial}{\partial u}\left(\frac{D''}{V}\right) - \frac{\partial}{\partial v}\left(\frac{D'}{V}\right) + \Gamma''\frac{D}{V} - 2\Gamma'\frac{D'}{V} + \Gamma\frac{D''}{V} &= 0. \end{aligned}\right\} \quad \ldots \quad (7)$$

These equations in the new independent variables are therefore necessary.

It will be noticed that the Monge-Ampere equation of deformation is obtained by substituting the values of D, D', D'' obtained from (5) in the Gaussian Equation (6).

$$K = \frac{\{x_{11} - x_1\Gamma - x_2\Delta\}\{x_{22} - x_1\Gamma'' - x_2\Delta''\} - \{x_{12} - x_1\Gamma' - x_2\Delta'\}^2}{X^2V^2} \quad \ldots \quad (8)$$

and
$$\begin{aligned} X^2V^2 &= (y_1z_2 - y_2z_1)^2 \\ &= (y_1^2 + z_1^2)(y_2^2 + z_2^2) - (y_1y_2 + z_1z_2)^2 \\ &= (E - x_1^2)(G - x_2^2) - (F - x_1x_2)^2 \\ &= V^2 - [Gx_1^2 - 2Fx_1x_2 + Ex_2^2]. \end{aligned}$$

This agrees with the result given in Forsyth's Differential Geometry, p. 363.

Substituting for D, D′, D″ from equations (5) in equation 7(1), we have

$$\frac{\partial}{\partial v}\left(\frac{x_{11}-x_1\Gamma-x_2\Delta}{VX}\right)-\frac{\partial}{\partial u}\left(\frac{x_{12}-x_1\Gamma'-x_2\Delta'}{VX}\right)$$

$$+\Delta''\left(\frac{x_{11}-x_1\Gamma-x_2\Delta}{VX}\right)-2\Delta'\left(\frac{x_{12}-x_1\Gamma'-x_2\Delta'}{VX}\right)$$

$$+\Delta\left(\frac{x_{22}-x_1\Gamma''-x_2\Delta''}{VX}\right)=0, \quad \ldots \quad (9)$$

$$\text{or, } \frac{1}{VX}\Big[x_{112}-x_{12}\Gamma-x_{22}\Delta-\Gamma_2x_1-x_2\Delta_2-x_{112}+x_{11}\Gamma'+x_{12}\Delta'$$

$$+x_1\Gamma'_1+x_2\Delta'_1+x_{11}\Delta''-2x_{12}\Delta'+x_{22}\cdot\Delta-\Delta''(x_1\Gamma+x_2\Delta)$$

$$+2\Delta'(x_1\Gamma'+x_2\Delta')-\Delta(x_1\Gamma''+x_2\Delta'')\Big]$$

$$-\frac{1}{V^2X^2}\left[x_{11}-x_1\Gamma-x_2\Delta\right]\frac{\partial(VX)}{\partial v}$$

$$+\frac{1}{V^2X^2}\left[x_{12}-x_1\Gamma'-x_2\Delta'\right]\frac{\partial(VX)}{\partial u}=0.$$

Co eff. of x_{11} $VX=\Gamma'+\Delta''-\dfrac{\partial \log VX}{\partial v}$

$$=\frac{\partial \log V}{\partial v}-\frac{\partial \log VX}{\partial v}=-\frac{\partial \log X}{\partial v}$$

[Eisenhart, Diff. Geom., Equation (3), Chapter V.]

Co-eff. of x_{12} $VX=-\Gamma-\Delta'+\dfrac{\partial \log (VX)}{\partial u}$

$$\frac{\partial \log V}{\partial u}+\frac{\partial \log VX}{\partial u}=-\frac{\partial \log X}{\partial u}.$$

Co-eff. of $x_1 . VX = -\Gamma_2 + \Gamma'_1 - \Delta''\Gamma + 2\Delta'\Gamma' - \Delta\Gamma''$

$$+\Gamma \frac{\partial \log VX}{\partial v} - \Gamma' \frac{\partial \log VX}{\partial u}$$

$$= -\Gamma_2 + \Gamma'_1 - \Delta''\Gamma + 2\Delta'\Gamma' - \Delta\Gamma''$$

$$+\Gamma \left\{ \frac{\partial \log X}{\partial v} + \Gamma' + \Delta'' \right\} - \Gamma' \left\{ \frac{\partial \log X}{\partial u} + \Gamma + \Delta' \right\}$$

$$= -\Gamma_2 + \Gamma'_1 + \Delta'\Gamma' - \Delta\Gamma'' + \Gamma \frac{\partial \log X}{\partial v} - \Gamma \frac{\partial \log X}{\partial u}$$

$$= F.K + \Gamma \frac{\partial \log X}{\partial v} - \Gamma' \frac{\partial \log X}{\partial u},$$

[Eisenhart, Equation (11), Chap. V]

using K to denote the right hand side expression of equation (6).

Co-eff. of $x_2 . VX = -\Delta_2 + \Delta'_1 - \Delta\Delta'' + 2\Delta'^2 - \Delta\Delta''$

$$+\Delta \frac{\partial \log VX}{\partial v} - \Delta' \frac{\partial \log VX}{\partial u}$$

$$= -\Delta_2 + \Delta'_1 + 2\Delta\Delta'' + 2\Delta'^2 + \Delta \left\{ \frac{\partial \log X}{\partial v} + \Gamma' + \Delta'' \right\}$$

$$- \Delta' \left\{ \frac{\partial \log X}{\partial u} + \Gamma + \Delta' \right\}$$

$$= -\Delta_2 + \Delta'_1 - \Delta\Delta'' + \Delta'^2 + \Delta\Gamma' - \Delta'\Gamma + \Delta \frac{\partial \log X}{\partial v} - \Delta' \frac{\partial \log X}{\partial u}$$

$$= -EK + \Delta \frac{\partial \log X}{\partial v} - \Delta' \frac{\partial \log X}{\partial u}.$$

Equation (9) therefore becomes

$$-\frac{\partial \log X}{\partial v} \left[x_{11} - \Gamma x_1 - \Delta x_2 \right] + \frac{\partial \log X}{\partial u} \left[x_{12} - \Gamma' x_1 - \Delta' x_2 \right]$$

$$+ K(F x_1 - E x_2) = 0. \qquad \ldots \quad (10)$$

We have seen that

$$X^2 = 1 - \frac{Ex_2^2 - 2Fx_1x_2 + Gx_1^2}{V^2} = 1 - \Phi(x),$$

$\Phi(x)$ standing for Beltrami's first differential parameter.

Similarly from the second Codazzi equation we obtain by the same substitution

$$-\frac{\partial \log X}{\partial v}\left[x_{12} - \Gamma' x_1 - \Delta' x_2\right] + \frac{\partial \log X}{\partial u}\left[x_{22} - \Gamma'' x_1 - \Delta'' x_2\right]$$

$$+ K(Gx_1 - Fx_2) = 0. \qquad \ldots \quad (11)$$

The three equations (6), (7) in the variables D, D', D'' or their equivalents (8), (10), (11) are therefore necessary.

To prove the sufficiency of the three equations (6), (7), we fall back on a theorem proved by Bonnet. He has shown that if six functions E, F, G, D, D', D'' of two variables u, v are given which satisfy the Gauss and the Codazzi equations, a surface can be determined unique except as regards its position and orientation in space having

$$Edu^2 + 2Fdu\,dv + Gdv^2 \text{ and } Ddu^2 + 2D'du\,dv + D''dv^2$$

as the first and the second fundamental quadrated forms respectively and the determination of the surface requires the integration of a Riccati equation and quadratures.

[Eisenhart, Diff. Geom., p. 159.]

It is clear, therefore, that any solution of equations (6) and (7) will determine a surface deformable into the given surface.

On the stability of two co-axial rectilinear vortices of compressible fluid

BY

BIBHUTIBHUSAN DATTA

The problem of the stability of the circular forms of two co-axial rectilinear vortices of incompressible fluid has been completely sloved by Poincare.[1] The object of the present paper is to extend his treatment to vortices of compressible fluid. The case, when the two vortices are outside each other, has been fully discussed by Dr. Chree[2] and by myself.

Taking the vortex to be parallel to the axis of z, the velocity components (u, v) at any point (x, y) in the fluid are given by[3]

$$u=-\frac{\partial \phi}{\partial x}-\frac{\partial \psi}{\partial y} \quad . \qquad \ldots \quad (1)$$

$$v=-\frac{\partial \phi}{\partial y}+\frac{\partial \psi}{\partial x} \quad ;$$

hence $$\nabla_1^2 \phi=-\theta. \quad \nabla_1^2 \psi=\zeta.$$

where $$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \quad \text{and} \quad \nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad .$$

Then by the theory of attraction.

$$\phi=-\frac{1}{2\pi}\iint \theta' \log r \, dx'dy',$$

$$\psi=\frac{1}{2\pi}\iint \zeta' \log r \, dx'dy', \qquad \ldots \quad (2)$$

[1] "*Théorie des Tourbillons*," Paris (1893) p. 165.

[2] "Vortices in a Compressible fluid." *Mess. Math.* Vol. XVII (1888), p. 105.

[3] Vide, Lamb's "*Hydrodynamics*" 4th Ed. (19[illegible]

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θ' and ζ' being respectively the values of θ and ζ at the point (x', y') and

$$r=\{(x-x')^2+(y-y')^2\}^{\frac{1}{2}};$$

the double integrations include all portions of the plane xy for which θ and ζ do not vanish.

The equation of continuity is

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\theta=0.$$

Therefore
$$\phi=\frac{1}{2\pi}\iint\frac{1}{\rho'}\frac{D\rho'}{Dt}\log r\,dx'dy'$$

Since θ and ζ are functions of distance from the centre we have, by the application of the First Mean Value Theorem, for circular vortex of cross section σ

$$\phi=\frac{1}{2\pi}\cdot\frac{\sigma}{\rho_0}\frac{D\rho_0}{Dt}\log r, \qquad \ldots \quad (3)$$

$$\psi=\frac{\zeta_0\sigma}{2\pi}\log r$$

where ρ_0 and ζ_0 are the mean values of the density and vorticity defined by the relations

$$\rho_0\sigma=\iint\rho'\,dx'dy', \quad \zeta_0\sigma=\iint\zeta'\,dx'dy'.$$

Again the equation of continuity is

$$\frac{D\rho_0\sigma}{Dt}=0,$$

or,
$$\frac{1}{\rho_0}\frac{D\rho_0}{Dt}+\frac{1}{\sigma}\frac{D\sigma}{Dt}=0.$$

Since σ always refers to a definite column of fluid, there is no difference between $\frac{D}{Dt}$ and the partial differential co-efficient $\frac{d}{dt}$.

Thus for a truly circular vortex of radius a and cross section σ, we have,

when $r>a$.

$$\phi_0=-\frac{1}{2\pi}\frac{d\sigma}{dt}\log r. \qquad \psi_0=\frac{k}{2\pi}\log r :$$

and when $r<a$.

$$\phi_0=-\frac{1}{4\pi a^2}\frac{d\sigma}{dt}r^2. \qquad \psi_0=\frac{k}{4\pi a^2}r^2.$$

where k is the strength of the vortex.

Consider two coaxial cylindrical vortices C and C', of radius a and b and of vorticity ζ and ζ' respectively. Suppose C to be inside C'. Now let the cross sections of the two tubes be slightly deformed so that at any instant their boundaries are given by

$$r=a+\Sigma(\alpha_n\cos n\theta+\beta_n\sin n\theta), \qquad \ldots \quad (4)$$

$$r=b+\Sigma(\alpha'_n\cos n\theta+\beta'_n\sin n\theta). \qquad \ldots \quad (5)$$

Then we shall have, from the continuity of ϕ and ψ, and of radial and tangential velocities at the surface, for C :

when $r<a$.

$$\psi_c=\text{const}+\frac{k}{4\pi a^2}r^2-\Sigma\frac{k}{2\pi n a}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\left(\frac{r}{a}\right)^n.$$

$$\phi_c=\text{const}-\frac{1}{4\pi a^2}\frac{d\sigma}{dt}r^2+\Sigma\frac{1}{2\pi n a}\frac{d\sigma}{dt}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\left(\frac{r}{a}\right)^n.$$

when $r>a$.

$$\psi'_c=\text{const}+\frac{k}{2\pi}\log r-\Sigma\frac{k}{2\pi n a}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\left(\frac{a}{r}\right)^n.$$

$$\phi'_c=\text{const}-\frac{1}{2\pi}\frac{d\sigma}{dt}\log r+\Sigma\frac{1}{2\pi n a}\frac{d\sigma}{dt}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\left(\frac{a}{r}\right)^n,$$

and for C' :

when $r<b$.

$$\psi_c'=\text{const}+\frac{k'}{4\pi b^2}r^2-\Sigma\frac{k'}{2\pi nb}(a'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{r}{b}\right)^n$$

$$\phi_c'=\text{const}-\frac{1}{4\pi b^2}\frac{d\sigma'}{dt}r^2+\Sigma\frac{1}{2\pi nb}\frac{d\sigma'}{dt}(a'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{r}{b}\right)^n,$$

when $r>b$.

$$\psi'_r=\text{const}+\frac{k'}{2\pi}\log r-\Sigma\frac{k'}{2\pi nb}(a'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{b}{r}\right)^n.$$

$$\phi'_r=\text{const}-\frac{1}{2\pi}\frac{d\sigma'}{dt}\log r+\Sigma\frac{1}{2\pi nb}\frac{d\sigma'}{dt}(a'_n\cos n\theta+\beta'_n\sin n\theta)\left(\frac{b}{r}\right)^n.$$

If $\mathfrak{R}$ denote the radial velocity of a point on the cylinder C and $n\Theta$ the velocity perpendicular to the radius vector, we have from (4),

$$\mathfrak{R}=\frac{da}{dt}+\Sigma\left(\frac{da_n}{dt}\cos n\theta+\frac{d\beta_n}{dt}\sin n\theta\right)-\Sigma n\Theta(a_n\sin n\theta-\beta_n\cos n\theta). \tag{6}$$

Since powers and products of the small quantities a_n, β_n are to be neglected we may take in (6),

$$\Theta=\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2} \qquad \ldots \quad (7)$$

We have als

$$\mathfrak{R}=-\frac{\partial\psi'_c}{r\partial\theta}-\frac{\partial\phi'_c}{\partial r}-\frac{\partial\psi_c'}{r\partial\theta}-\frac{\partial\phi_c}{\partial r}-\frac{a}{2\pi b^2}\frac{d\sigma'}{dt}. \qquad \ldots \quad (8)$$

where r is to be put equal to $a+\Sigma(a_n\cos n\theta+\beta_n\sin n\theta)$ after differentiation

The two values of $\mathfrak{R}$ given by (6) and (8) must be identical. The terms independent of θ simply verify the known result $\frac{d\sigma}{dt}=2\pi a$

$\times \frac{da}{dt}$. Equating the co-efficients of $\cos n\theta$ and $\sin n\theta$, we get

$$\left.\begin{aligned}
\frac{da_n}{dt} &= -n\left(\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2}\right)\beta_n+\frac{k}{2\pi a^2}\beta_n+\frac{k'}{2\pi b^2}\epsilon^{n-1}\beta'_n \\
&\quad +\frac{1}{2\pi b^2}\frac{d\sigma'}{dt}a_n-\frac{\epsilon^{n-1}}{2\pi b^2}\frac{d\sigma'}{dt}a'_n; \\
\frac{d\beta_n}{dt} &= n\left(\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2}\right)a_n-\frac{k}{2\pi a^2}a_n-\frac{k'}{2\pi b^2}\epsilon^{n-1}a'_n \\
&\quad +\frac{1}{2\pi b^2}\frac{d\sigma'}{dt}\beta_n-\frac{\epsilon^{n-1}}{2\pi b^2}\frac{d\sigma'}{dt}\beta'_n;
\end{aligned}\right\} \quad (9)$$

where $\epsilon=a/b$. Similarly from the condition to be satisfied at the surface of the vortex C' we get the equations:

$$\left.\begin{aligned}
\frac{da'_n}{dt} &= -n\left(\frac{k}{2\pi b^2}+\frac{k'}{2\pi b^2}\right)\beta'_n+\frac{k}{2\pi a^2}\epsilon^{n+1}\beta_n+\frac{k'}{2\pi b^2}\beta'_n \\
&\quad -\frac{1}{2\pi b^2}\frac{d\sigma}{dt}a'_n+\frac{\epsilon^{n+1}}{2\pi a^2}\frac{d\sigma}{dt}a_n, \\
\frac{d\beta'_n}{dt} &= n\left(\frac{k}{2\pi b^2}+\frac{k'}{2\pi b^2}\right)a'_n-\frac{k}{2\pi a^2}\epsilon^{n+1}a_n-\frac{k'}{2\pi b^2}a'_n \\
&\quad -\frac{1}{2\pi b^2}\frac{d\sigma}{dt}\beta'_n+\frac{\epsilon^{n+1}}{2\pi a^2}\frac{d\sigma}{dt}\beta_n.
\end{aligned}\right\} \quad (10)$$

For abbreviation, put

$$\left.\begin{aligned}
f &= \frac{k}{2\pi a^2}-n\left(\frac{k}{2\pi a^2}+\frac{k'}{2\pi b^2}\right), \quad g=\frac{k'}{2\pi b^2}\epsilon^{n-1}, \quad h=\frac{1}{2\pi b^2}\frac{d\sigma}{dt}, \\
f' &= \frac{k'}{2\pi b^2}-n\left(\frac{k}{2\pi b^2}+\frac{k'}{2\pi b^2}\right), \quad g'=\frac{k}{2\pi a^2}\epsilon^{n+1}, \quad h'=\frac{1}{2\pi b^2}\frac{d\sigma'}{dt}.
\end{aligned}\right\} \quad (11)$$

The equations (9) and (10) can, then, be written as

$$\left.\begin{aligned}
\frac{da_n}{dt} &= f\beta_n + g\beta'_n + h'(a_n - \epsilon^{n-1}a'_n),\\
\frac{d\beta_n}{dt} &= -fa_n - g\alpha'_n + h'(\beta_n - \epsilon^{n-1}\beta'_n),\\
\frac{da'_n}{dt} &= f'\beta'_n + g'\beta_n + h(\epsilon^{n-1}a_n - a'_n),\\
\frac{d\beta'_n}{dt} &= -f'a'_n - g'a_n + h(\epsilon^{n-1}\beta_n - \beta'_n).
\end{aligned}\right\} \quad \ldots (12)$$

Denote

$$\left.\begin{aligned} u &= a_n + i\beta_n, \\ v &= a_n - i\beta_n; \end{aligned}\right\} \quad \left.\begin{aligned} u' &= a'_n + i\beta'_n, \\ v' &= a'_n - i\beta'_n; \end{aligned}\right\} \quad \ldots (13)$$

where $$i = \sqrt{-1}.$$

Then from equations (12), we get

$$\left.\begin{aligned}
\frac{du}{dt} - (h' - if)u &= -(h'\epsilon^{n-1} + ig)u',\\
\frac{dv}{dt} - (h' + if)v &= -(h'\epsilon^{n-1} - ig)v',\\
\frac{du'}{dt} + (h + if')u' &= (h\epsilon^{n-1} - ig')u,\\
\frac{dv'}{dt} + (h - if')v' &= (h\epsilon^{n-1} + ig')v.
\end{aligned}\right\} \quad \ldots (14)$$

Eliminating u' between the first and the third of these equations we get

$$\frac{d}{dt}\left[\frac{e^{\int(h+if')dt}}{h'\epsilon^{n-1}+ig}\left\{\frac{du}{dt} - (h' - if)u\right\}\right] + (h\epsilon^{n-1} - ig')ue^{\int(h+if')dt} = 0;$$

and similarly we obtain the differential equations for u', v and v' as

$$\frac{d}{dt}\left[\frac{e^{\int(h-if')dt}}{h'\epsilon^{n-1}-ig}\left\{\frac{dv}{dt}-(h'+if')v\right\}\right]+(h\epsilon^{n-1}+ig')ve^{\int(h-if')dt}=0,$$

$$\frac{d}{dt}\left[\frac{e^{-\int(h'-if)\,dt}}{h\epsilon^{n-1}-ig'}\left\{\frac{du'}{dt}+(h+if')\,u'\right\}\right]$$

$$+(h'\epsilon^{n-1}+ig)u'e^{-\int(h'-if)dt}=0.$$

$$\frac{d}{dt}\left[\frac{e^{-\int(h'+if)dt}}{h\epsilon^{n-1}+ig'}\left\{\frac{dv'}{dt}+(h-if')v'\right\}\right]$$

$$+(h'\epsilon^{n-1}-ig)v'e^{-\int(h'+if)dt}=0.$$

Simplifying we get the differential equations for the determination of u, u', v, v' as

$$\frac{d^2u}{dt^2}+P\frac{du}{dt}+Qu=0,$$

$$\frac{d^2v}{dt^2}+R\frac{dv}{dt}+Sv=0. \qquad \ldots \quad (15)$$

$$\frac{d^2u'}{dt^2}+P'\frac{du'}{dt}\,Q'u'=0.$$

$$\frac{d^2v'}{dt^2}+R'\frac{dv'}{dt}+S'v'=0;$$

where

$$P=(h+if')-(h'-if)-\frac{d}{dt}(h'\epsilon^{n-1}+ig)\div(h'\epsilon^{n-1}+ig),$$

$$P'=(h+if')-(h'-if)\ -\frac{d}{dt}(h\epsilon^{n-1}-ig')\div(h\epsilon^{n-1}-ig')$$

$$R=(h-if')-(h'+if)-\frac{d}{dt}(h'\epsilon^{n-1}-ig)\div(h'\epsilon^{n-1}-ig)$$

$$R'=(h-if')-(h'+if)-\frac{d}{dt}(h'\epsilon^{n-1}+ig)\div(h'\epsilon^{n-1}+ig')$$

$$Q=(he^{n-1}-ig')(h'e^{n-1}+ig)-\frac{d}{dt}(h'-if)$$
$$-(h'-if)\{(h+if')-\frac{d}{dt}(h'e^{n-1}+ig)/(h'e^{n-1}+ig)\}$$

$$Q'=(h'e^{n-1}+ig)(he^{n-1}-ig')+\frac{d}{dt}(h+if')$$
$$-(h+if')\{(h'-if)+\frac{d}{dt}(he^{n-1}-ig')/(he^{n-1}-ig')\},$$

$$S=(h'e^{n-1}-ig')(h'e^{n-1}+ig)-\frac{d}{dt}(h'+if)$$
$$-(h'+if)\{(h-if')-\frac{d}{dt}(h'e^{n-1}-ig)/(h'e^{n-1}-ig)\},$$

$$S'=(h'e^{n-1}-ig)(he^{n-1}+ig')+\frac{d}{dt}(h'-if')$$
$$-(h-if')\{(h'+if)+\frac{d}{dt}(he^{n-1}-ig')/(he^{n-1}-ig')\} \quad (16)$$

These are equations of second degree with variable co-efficients and at the present state of mathematical analysis they cannot be generally solved. But that will not deter us from the discussion of our problem. For the necessary and sufficient condition for the stability of the circular from is that α's and β's, therefore u, u'...must be expressible in the form[1]

$$e^{itp}$$

where p is a real quantity.

Now take the first of the equations (15), *viz.*,

$$\frac{d^2u}{dt^2}+P\frac{du}{dt}+Qu=0.$$

This can be transformed into the form

$$\frac{d^2v}{dt^2}+Iv=0. \quad \ldots \quad (17)$$

[1] Poincaré, l.c. p. 109.

where

$$u=re^{-\frac{1}{2}\int P dt}, \qquad \ldots \quad (18)$$

and

$$1=Q-\tfrac{1}{2}\frac{dP}{dt}-\tfrac{1}{4}P^2. \qquad \ldots \quad (19)$$

Now P is a complex quantity of the from $\beta+i\gamma$; hence we conclude that the motion is unstable.

Put $\alpha=\alpha_n=\beta_n=0$; so that there is only a single compressible vortex of radius b, then, from (11)

$$f=-\frac{k'}{2\pi b^2}\,;\ f'=(1-n)\frac{k'}{2\pi b^2}\,.\ h'=\frac{1}{2\pi b^2}\frac{d\sigma'}{dt}\,; \qquad \ldots \quad (20)$$

$$e=g=h=g'=0.$$

The equations (14) then reduce to

$$\frac{da'}{dt}-i(n-1)\frac{k'}{2\pi b^2}a'=0,$$

$$\frac{d\sigma'}{dt}+i(n-1)\frac{k'}{2\pi b^2}\sigma'=0;$$

whence

$$a'=A\ \exp.\left(i(n-1)\int_0^t\frac{k'}{2\pi b^2}\,dt\right). \qquad \ldots \quad (21)$$

$$\sigma'=A\ \exp.\left(-i(n-1)\int_0^t\frac{k'}{2\pi b^2}\,dt\right).$$

where A is a constant of integration.

Therefore

$$\alpha'_n=A\cos\left\{k'(n-1)\int\frac{dt}{\sigma'}\right\}. \qquad \ldots \quad (22)$$

$$\beta'_n=A\sin\left\{k'(n-1)\int_0^t\frac{dt}{\sigma'}\right\}.$$

which agree with the results of Dr. Chree.[1]

[1] *l.c.*, p. 112.

Again let $h=h'=0$; so that there are two co-axial vortices of incompressible fluid. In this case f, f', g, g'...are independent of time; then

$$P=P'=i(f+f') \; ; \qquad R=R'=-i(f+f'), \qquad \ldots \quad (23)$$

$$Q=Q'=gg'-ff'. \qquad S=S'=ff''-gg'.$$

The first equation of (15) becomes

$$\frac{d^2u}{dt^2}+i(f+f')\frac{du}{dt}+(gg'-ff')u=0. \qquad \ldots \quad (24)$$

Assume $u=e^{iat}$.

Substituting in the previous equation, we get the equation for a as

$$a^2+(f+f')a-(gg'-ff')=0. \qquad \ldots \quad (25)$$

The necessary and sufficient condition of stability is that a must be real; therefore

$$(f+f')^2+4(gg'-ff')>0,$$

or

$$(f-f')^2+4gg'>0.$$

substituting the values of f, f', g, g' from (11), get

$$(\zeta-u(\zeta+\zeta'))(-\zeta'+u(\zeta\epsilon^2+\zeta'))^2+4\zeta\zeta'\epsilon^2>0 \qquad \ldots \quad (26)$$

which is the condition obtained by Poincaré.[1]

[1] *l.c.*, p. 172.

On some Properties of Natural Numbers

BY

HARIPADA DATTA

Here in this note a few theorems in natural numbers have been proved. The truth of some of these theorems may be perceived by merely observing the number when written in order of magnitude.

Let N, M and P be three positive integers and $N=M+P$. And let us express them in the scale of another positive integer r.

$$N=a_n r^n+a_{n-1}r^{n-1}+\ldots+a_1 r+a_0.$$

$$M=b_n r^n+\ldots+b_1 r+b_0.$$

$$P=c_n r^n+\ldots+c_1 r+c_0.$$

and

$$\phi(N)=na_n r^n+(n-1)a_{n-1}r^{n-1}+\ldots+a_1 r.$$

The co-efficients a, b, c are all less than r.

I. $$\Sigma a \not> \Sigma b+\Sigma c.$$

i.e., $$\not> \Sigma(b+c).$$

If $\Sigma a>\Sigma(b+c)$, then at least one of the a's, say a_s, will be greater than (b_s+c_s) and the first $(s-1)$ a's are equal to the corresponding $(b+c)$'s.

$$a_n=b_n+c_n \ldots\ldots a_{s-1}=b_{s-1}+c_{s-1};$$

$$a_s \neq b_s+c_s.$$

We have

$$a_n r^n+\ldots+a_s r^s+\ldots+a_0$$

$$=(b_n+c_n)r^n+\ldots+(b_s+c_s)r^s+\ldots+(b_0+c_0).$$

Now by re-arranging the terms on the right of $(b_s+c_s)r^s$, we obtain not only all the terms on the right of $a_s r^s$ but also $(a_s-b_s-c_s)\,r^s$; for $(b_s+c_s)\,r^s+(a_s-b_s-c_s)\,r^s=a_s r^s$. $(a_s-b_s-c_s)\,r^s$ has been obtained as the sum of parts of some or all the terms following $a_s r^s$. This sum is the sum of parts of co-efficients multiplied by quantities less than r^s. Hence this sum of the co-efficients is greater than $(a_s-b_s-c_s)$. Therefore

$$\Sigma a<\Sigma(b+c).$$

From this it follows that

(i) $a_s = b_s + c_s + 1$.

(ii) if any of the a's is not equal to the corresponding $(b+c)$ then $\Sigma a < \Sigma (b+c)$; and *vice-versa*.

(iii) if $\Sigma a = \Sigma (b+c)$ then the a's are separately equal to the corresponding $(b+c)$'s.

II. $$\phi(N) \leqslant \phi(M) + \phi(P).$$

Suppose $$a_u = b_u + c_u ; \ldots a_{s+1} = b_{s+1} + c_{s+1},$$

and $$a_s \neq b_s + c_s.$$

As a_s is greater than $(b_s + c_s)$ by unity, so there must be some co-efficient of $(M+P)$, say $(b_p + c_p)$, which will be greater than a_p by r; that is, $a_p + r = b_p + c_p$, and the co-efficients of $(M+P)$ between $(b_s + c_s)$ and $(b_p + c_p)$ must be

$$(a_{p-1} + r - 1), (a_{p-2} + r - 1), \ldots (a_{p+1} + r - 1).$$

If $$\phi(N) = A(r) + s a_s r^s + B(r),$$

then $$\phi(M) + \phi(P) = A(r) + s(a_s - 1) r^s + B(r)$$

$$+ \left\{ (s-1) r^s - r \frac{r^{s-1} - 1}{r-1} \right\} - \left\{ (p-1) r^p - r \frac{r^{p-1} - 1}{r-1} \right\} + p r^p.$$

Now $$\phi(N) > \phi(M) + \phi(P).$$

if $$s a_s r^s > \left(s a_s - 1 - \frac{1}{r-1} \right) r^s + \left(1 + \frac{1}{r-1} \right) r^p. \quad \ldots \quad (x)$$

i.e., if $$\left(1 + \frac{1}{r-1} \right) r^s > \left(1 + \frac{1}{r-1} \right) r^p.$$

(i) As $s > p$, hence $\phi(N) > \phi(M) + \phi(P)$.

(ii) *If all the a's are separately equal to the corresponding $(b+c)$'s, then*

$$\phi(N) = \phi(M) + \phi(P).$$

(iii) From (x) we see that the difference

$$\phi(N) - \{\phi(M) + \phi(P)\}$$

$$= \left(1 + \frac{1}{r-1} \right) r^s - \left(1 + \frac{1}{r-1} \right) r^p$$

which is a positive quantity.

As s increases and p decreases, the difference increases. The larger or the more numerous are the gaps the greater is the difference. That is to say as $\Sigma(b+c)$ increases the value of $\{\phi(M)+\phi(P)\}$ decreases. The greatest value of $\Sigma(b+c)$

$$=(a_n-1)+(a_{n-1}+r-1)+\ldots+(a_1+r-1)+(a_0+r)$$

$$=\Sigma a+n(r-1).$$

Therefore *the least value of* $\{\phi(M)+\phi(P)\}$

$$=\phi(N)-\left(1+\frac{1}{r-1}\right)r^n+\frac{r}{r-1}.$$

III. The factors of natural numbers, when written in order of magnitude, which are of the form r^s (where $s=1,2,\ldots$), are

$$\left.\begin{array}{llll} r, & r, & \ldots & r, \; r^2 \\ r, & r, & \ldots & r, \; r^2 \\ \ldots & \ldots & & \ldots \\ r, & r, & \ldots & r, \; r^2 \\ r, & r, & \ldots & r, \; r^3 \\ r, & r, & \ldots & r, \; r^2 \\ \ldots & \ldots & & \ldots \end{array}\right\} \quad \ldots \quad (\alpha)$$

The first row contains the first r—factors, the second row contains the second r—factors and so on. The first $(r-1)$ factors of all the rows are r. The last factors of the first $(r-1)$ rows are r^2. The last factor of the rth row is r^3. Of the (r^2-1) rows the last factor of every rth row, is r^3, and the last factors of all the other rows are r^2. The last factor of the r^2th row is r^4.

The sum of the powers of r up to the first r is $\frac{r-1}{r-1}$.

... ... r^2 is $\frac{r^2-1}{r-1}$,

... ... r^3 is $\frac{r^3-1}{r-1}$

(i) *In the product of the first N factors, the powers of r*

$$=a_n\frac{r^{n+1}-1}{r-1}+a_{n-1}\frac{r^n-1}{r-1}+\ldots+a_0\frac{r-1}{r-1}$$

$$=\frac{r}{r-1}\mathrm{N}-\frac{1}{r-1}\Sigma a.$$

In the joint product of the first M and the first P *factors the powers of r*

$$=\frac{r}{r-1}(\mathrm{M}+\mathrm{P})-\frac{1}{r-1}\Sigma(b+c).$$

As $\Sigma a \leqslant \Sigma(b+c)$, therefore ,

(ii) *the product of the first M factors is either equal to or less than the product of any other M consecutive factors.*

(iii) if r be a prime number, then at once follows the known theorem, namely, *the product of M consecutive numbers is divisible by M !* That is, $\binom{x}{M}$ is an integer, where x is the first term of the M consecutive numbers.

(iv) *If the highest power of r in the M consecutive factors be greater than the highest power of r in the first M factors, then the product of the M consecutive factors will be greater than the product of the first M factors.*

Let $$x=u_s r^s+u_{s-1}r^{s-1}+\ldots+u_1 r+u_0$$

and $$x+\mathrm{M}=v_m r^m+\ldots+v_1 r+v_0$$

Suppose $$u_s r^s=v_m r^m,\ldots u_{s-w-1}r^{s-w-1}=v_{s-w-1}r^{m-w-1},$$

and $$u_{s-w}r^{s-w}\neq v_{s-w}r^{m-w},$$

$$v_{s-w}r^{m-w}\leqslant b_n r^n.$$

If $v_{s-w}r^{m-w}>b_n r^n$, *then the integer* $\binom{x}{M}$ *is divisible by r.*

(v) *If* $\mathrm{M}+\mathrm{P}=\mathrm{N}-1$, *then*

$$\frac{r}{r-1}\mathrm{N}-\frac{1}{r-1}\Sigma a<\frac{r}{r-1}(\mathrm{M}+\mathrm{P})-\frac{1}{r-1}\Sigma(b+c)+(n+1).$$

Therefore $$\Sigma(b+c)-\Sigma a<(r-1)(n+1).$$

(vi) More generally, *to find the product of the factors of the form* $r_1^{x_1} r_2^{x_2} \dots r_n^{x_n}$ *which is contained in* $n!$, *where the* r's *are prime numbers,*

Let the product be $\binom{x_1+1}{r_1}\binom{x_2+1}{r_2}\cdots$ and the integral part of $\frac{r}{r_1 r_2 \dots r_n}$ be m.

Let
$$\begin{aligned} m &= a_1 r_1^{s_1} + a_2 r_1^{s_2} + \dots + a_u r_1^{s_u} \\ &= b_1 r_2^{t_1} + b_2 r_2^{t_2} + \dots + b_v r_2^{t_v} \\ &= c_1 r_3^{k_1} + c_2 r_3^{k_2} + \dots + c_w r_3^{k_w} \\ &\dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

Hence
$$a_1 = \frac{r_1}{r_1-1} m - \frac{1}{r_1-1}(a_1+a_2+\dots+a_u)$$
$$a_2 = \frac{r_2}{r_2-1} m - \frac{1}{r_2-1}(b_1+b_2+\dots+b_v)$$
$$a_3 = \frac{r_3}{r_3-1} m - \frac{1}{r_3-1}(c_1+c_2+\dots+c_w)$$
$$\dots \quad \dots \quad \dots \quad \dots$$

The sum of these factors
$$= \frac{m(m+1)}{2} r_1 r_2 \dots r_n.$$

IV. From (a) III, we have

The sum of the factors up to the first $r=r_1$

" " $r_1^2 = r^2 + (r-1)r$,

" " $r_1^3 = r^3 + 2(r-1)r^2$.

$\dots \quad \dots \quad \dots \quad \dots \quad \dots$

" " $r^n = r^n + (n-1)(r-1)r^{n-1}$.

(i) *The sum of the first* N *factors of* (a) *III.*
$$= rN + (r-1)\phi(N).$$

The sum of the first M *factors*

$$=rM+(r-1)\phi(M).$$

The sum of the first P *factors*

$$=rP+(r-1)\phi(P).$$

(ii) As $\phi(N)\leq\phi(M)+\phi(P)$,

hence *the sum of any M consecutive factors is either equal to or greater than the sum of the first M factors.*

V. *To find the product of all the numbers which are of the form* $r_1^{p_1} r_2^{p_2}$ *and less than or equal to a given number N. Neither* p_1 *nor* p_2 *is zero and both* r_1 *and* r_2 *are primes.*

Let $r_1<r_2$; and the integral part of $\frac{N}{r_1}$

$$=b_1r_2^{s}+b_2r_2^{s-1}+\ldots+b_sr_2+b_{s+1}$$

And

$$b_1=c_1r_1^{x_1-1}+\ldots.$$

$$b_1r_2+b_2=c_2r_1^{x_2-1}+\ldots.$$

$$b_1r_2^{2}+b_2r+b_3=c_3r_1^{x_3-1}+\ldots.$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$b_1r_2^{s-1}+\ldots+b_s=c_sr_1^{x_s-1}+\ldots.$$

The highest number which contains r_2^{s} *is* $r_2^{s}r_1^{x_1}$

for $$r_2^{s}r_1^{x_1}<r_2^{s}(c_1r_1^{x_1-1}+\ldots)r_1$$

$$<b_1r_2^{s}r_1$$

$$<n\ ;$$

but $$r_2^{s}r_1^{x_1+1}\geq r_1\{(b_1+1)r_2^{s}+\ldots\}$$

$$>r_1\frac{n}{r_1}$$

$$>n.$$

The highest number which contains r_2^{s-1} is $r_2^{s-1}r_1^{x_2}$

for
$$r_2^{s-1}r_1^{x_2} < \left\{r_2 r_1^{x_1-1} + \dots\right\} r_2^{s-1}$$
$$< r_1(b_1 r_1 + b_2) r_2^{s-1}$$
$$< n;$$

but
$$r_2^{s-1}r_2^{x_2+1} \geq r_1 r_2^{s-1}(b_1 r_1 + b_2 + 1)$$
$$> r_1 \frac{n}{r_1}$$
$$> n.$$

In this way we can show that the product

$$= \left\{r_2^{x_1 s} \times r_1^{\frac{1}{2}r_1(x_1+1)}\right\} \left\{r_2^{x_2(s-1)} r_1^{\frac{1}{2}x_2(x_2+1)}\right\}$$
$$\dots \left\{r_2^{x_s} r_1^{\frac{1}{2}x_s(x_s+1)}\right\}.$$

And the number of these numbers $= (x_1 + x_2 + \dots + x_s)$.

And their sum

$$= r_2^s \frac{r_1^{x_1+1}-1}{r_1-1} + r_2^{(s-1)} \frac{r_1^{x_2+1}-1}{r_1-1} + \dots + r_2^{r_1^{x_s+1}-1}.$$

For practical purposes, in finding the values of x's, the following method may be adopted with advantage over the other.

If
$$r_2 = b_1 r_1^p + \dots.$$

then the highest term containing
$$r_2^s = r_2^s r_1^{x_1+1} \quad \text{and} \quad r_2^{s-1} \nmid r_2^{s-1} r_1^{x_1+p+1}$$

for
$$r_2^{s-1} r_1^{x_1+p+1} < r_2^s r_1^{x_1+1}.$$

Similarly the highest term containing

$$r_2^{s-2} \not< r_2^{s-2} r_1^{x_1+2p+1} \quad \text{and} \quad r_3^{s-3} \not< r_3^{s-3} r_1^{x_1+3p+1}$$

and so on.

Now multiplying

$$r_2^{s-1} r_1^{x_1+p+1} \quad \text{by} \quad r_1^{D} \quad (D=0,1,\ldots),$$

we can find x_2. In this way we can find the values of other x's.

VI. *To find the product of all numbers of the form* $r_1^{p_1} r_2^{p_2} r_3^{p_3}$ *and less than or equal to a given number* N, where r_1, r_2, and r_3 are prime numbers and none of the p's are zero.

Let the integral part of $\frac{N}{r_1 r_2 r_3}$ be M_1. The product of all the numbers of the form $r_1^{p_1} r_2^{p_2}$, $r_1^{p_1} r_3^{p_3}$ and $r_2^{p_2} r_3^{p_3}$ and not greater than M_1, are respectively

$$\left\{ r_2^{x_1} r_1^{\frac{1}{2}x_1(x_1+1)} \right\} \cdots \left\{ r_2^{x_s} r_1^{\frac{1}{2}x_s(x_s+1)} \right\} \qquad \ldots \quad (1)$$

$$\left\{ r_3^{y_1} r_1^{\frac{1}{2}y_1(y_1+1)} \right\} \cdots \left\{ r_3^{y_t} r_1^{\frac{1}{2}y_t(y_t+1)} \right\} \qquad \ldots \quad (2)$$

and

$$\left\{ r_3^{z_1} r_2^{\frac{1}{2}z_1(z_1+1)} \right\} \cdots \left\{ r_3^{z_u} r_2^{\frac{1}{2}z_u(z_u+1)} \right\}. \qquad \ldots \quad (3)$$

Let

$$M_1 = \alpha r_1^{\theta} + \ldots$$

$$= \beta r_2^{\sigma} + \ldots$$

$$= \gamma r_3^{\delta} + \ldots.$$

Thus we get the numbers

$$r_1 r_2 r_3 r_1^{\theta(\theta+1)} \quad (4)$$

$$r_1 r_2 r_3 r_2^{\frac{\sigma}{2}(\sigma+1)} , \quad \dots (5)$$

and

$$r_1 r_2 r_3 r_3^{\frac{\delta}{2}(\delta+1)} . \quad \dots (6)$$

Let us denote the combined product of (1), (2), (3), (4), (5) and (6) by $r_1 r_2 r_3 \psi(M_1)$.

Of these M_1 numbers, there are numbers which contain $r_1 r_2 r_3$ as factors.

Let the integral part of $\frac{M_1}{r_1 r_2 r_3}$ be M_2. Here we obtain the product $(r_1 r_2 r_3)^2 \psi(M_2)$. Continuing this process we shall come to a number M_{m+1} which will be less than $r_1 r_2 r_3$.

Thus the required product

$$= r_1 r_2 r_3 \psi(M_1)(r_1 r_2 r_3)^2 \psi(M_2) \dots \dots (r_1 r_2 r_3)^m \psi(M_m)(r_1 r_2 r_3)^{1+2+\dots+m}$$

$$= \psi(M_1)\psi(M_2)\dots\psi(M_m)(r_1 r_2 r_3)^{\frac{m(m+1)}{2}} .$$

As for example let us find the product of all the numbers of the form $2.^{p_1} 3.^{p_2} 5^{p_3}$ and less than or equal to 9000.

Here

$$\frac{9000}{2.3.5} = 300.$$

$$\frac{300}{3} = 100 = 2^6 + \dots,$$

$$\frac{300}{2} = 150 = 2^4 + \dots.$$

Hence the numbers of the form $2.^{p_1}3.^{p_2}2.3.5$ and not greater than 9000, are

(1) $3^{\alpha}2.2.3.5$; $3^{\beta}\{2.2^{\alpha}.2^{\beta}\}^{*}2.3.5$; $3^{\gamma}\{2.2^{\alpha}.2^{\beta}.2^{\gamma}.2^{\delta}\}2.3.5$

and $3\{2.2^{\alpha}.2^{\beta}.2^{\gamma}.2^{\delta}.2^{\epsilon}\}2.3.5.$

Those which are of the form $2.^{p_1}5.^{p_2}2.3.5$ are

(2) $5^{\alpha}.2.2.3.5$; $5^{\beta}\{2.2^{\alpha}.2^{\beta}\}2.3.5$; $5\{2.2^{\alpha}.2^{\beta}.2^{\gamma}.2^{\delta}\}2.3.5.$

Those of the form $3.^{p_1}5.^{p_2}2.3.5$ are

(3) $5^{\alpha}\{3.3^{\alpha}\}2.3.5$; $5\{3.3^{\alpha}.3^{\beta}\}2.3.5.$

The numbers of the form $\left\{2,^{p_1}3,^{p_2}5^{p_3}\right\}2.3.5$ respectively are

(4) $\{2.2^{\alpha}.2^{\beta}2^{\gamma}.2^{\delta}.2^{\epsilon}\}2.3.5,$

(5) $\{3.3^{\alpha}.3^{\beta}.3^{\gamma}\}2.3.5,$

(6) $\{5.5^{\alpha}.5^{\beta}\}2.3.5.$

Again

$$\frac{300}{2.3.5}=10,\ \frac{10}{3}=2+1;\ \frac{10}{2}=3+2.$$

Hence in addition to the above we have also the following numbers

(7) $3.2.(2.3.5)^{\alpha}$

(8) $5.2.(2.3.5)^{\alpha}$

(9) $\{2.2^{\alpha}.2^{\beta}\}(2.3.5)^{\alpha}$

(10) $\{3.3^{\alpha}\}(2.3.5)^{\alpha}$

(11) $5.(2.3.5)^{\alpha}$

(12) $2.3.5$; $(2.3.5)^{\alpha}$.

The number of these numbers is 52; the product of these 52 numbers is $2.^{149}3.^{107}5.^{84}$.

* $[a\{x.y.z\}b]=abx.aby.abz.$

A Note on Whittaker's Formula for the Solution of Algebraic or Transcendental Equations

BY

BHOLANATH PAL

In a recent paper, published in the Proceedings of the Edinburgh Mathematical Society (November, 1918), Prof. Whittaker has communicated the following elegant formula for obtaining the root, which is the smallest in absolute value, of algebraic or transcendental equations. He has shown that the root of the equation

$$0=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$$

which is the smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2}{a_1^3}\frac{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}-\frac{a_0^3}{a_1^4}\frac{\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}-\frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}-\frac{a_0^4}{a_1^5}\frac{\begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}-\frac{\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}-\ldots \quad (1)$$

This series converges rapidly when the ratio of the smallest root to every one of the other roots is small.

In the following note I propose to show that the method given by Whittaker can be used to determine the absolute values of all the real roots of an algebraic or transcendental equation. Supposing x_0 to be the smallest root and x_1 to be the next higher root we can write $x_1 = x_0 + h$, where h has to be determined. Substituting $x_0 + h$ for x in the above equation and writing $f(x)$ for the expression

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

and then expanding by Taylor's theorem, we obtain

$$f'(x_0) + \frac{1}{|2} f''(x_0) h + \frac{1}{|3} f'''(x_0) h^2 + \frac{1}{|4} f^{iv}(x_0) h^3 + \frac{1}{|5} f^{v}(x_0) h^4 + \ldots = 0$$

as an equation in h. By Whittaker's formula the value of h, which is the smallest root in absolute value, of this equation is given by the series

$$-\frac{f'(x_0)}{\left|\begin{matrix} \frac{1}{|2} f''(x_0) & \left|\begin{matrix} \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) \\ f'(x_0) & \frac{1}{|2} f''(x_0) \end{matrix}\right| \\ \left\{f'(x_0)\right\}^2 & \left|\begin{matrix} \frac{1}{|3} f'''(x_0) & \frac{1}{|4} f^{iv}(x_0) \\ \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) \end{matrix}\right| \end{matrix}\right|}{\left|\begin{matrix} \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) & \left|\begin{matrix} \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) & \frac{1}{|4} f^{iv}(x_0) \\ f'(x_0) & \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) \\ 0 & f'(x_0) & \frac{1}{|2} f''(x_0) \end{matrix}\right| \\ f'(x_0) & \frac{1}{|2} f''(x_0) & \left|\begin{matrix} \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) & \frac{1}{|4} f^{iv}(x_0) \\ f'(x_0) & \frac{1}{|2} f''(x_0) & \frac{1}{|3} f'''(x_0) \\ 0 & f'(x_0) & \frac{1}{|2} f''(x_0) \end{matrix}\right| \end{matrix}\right|} - \ldots \quad (2)$$

where $f'(x_0), f''(x_0) \ldots$, mean $\frac{df(x_0)}{dx_0}, \frac{d^2 f(x_0)}{dx_0^2}, \ldots$.

Therefore the second real root of the original equation is given by

$$\theta_1 = \text{series (1)} + \text{series (2)}.$$

This process can obviously be extended to obtain all the higher roots one by one.

Since the value of the smallest root has to be obtained from an infinite series the determination of this root will always involve a certain amount of error. This error will be carried on in the determination of the next higher root and so on. So that after a certain stage this method will fail to give the value of the higher roots to any desired degree of approximation. This defect in the method can be partially avoided by replacing the equation in its original stage as well as in the different transformed stages for the determination of the higher roots by one which has undergone two or three Lobatchevsky-Graeffe operations, each of which replaces the equation operated on by an equation whose roots are the squares of its roots, so as to make the series for the determination of the roots more and more convergent and thereby minimise the error as far as possible in each step of the process.

As an example of this method, I have worked out the first three roots of the well known equation in Mathematical Physics:

$$\theta = \tan\theta,$$

$$\frac{2\theta^3}{3} - \frac{4\theta^5}{5} + \frac{6\theta^7}{7} - \frac{8\theta^9}{9} + \frac{10\theta^{11}}{11} - \ldots = 0,$$

or

$$1 - \cdot 1\theta^2 + \cdot 0033571\theta^4 - \cdot 0000661 3\theta^6 + \cdot 0000007 5\theta^8 - \ldots = 0.$$

Considering this as an equation in θ^2, and using Whittaker's formula we get the smallest root of the equation in the form

$$\theta_0^2 = 10 + 5\cdot 5623 + 2\cdot 7727 + 1\cdot 1428 + \cdot 4834 + \ldots.$$

Taking sufficient number of terms and retaining only to the second place of decimals, we get

$$\theta_0^2 = 20\cdot 20.$$

Therefore,

$$\theta_0 = \pm 4\cdot 49.$$

Then we seek the next higher root by the present method. The equation in h is

$$-\cdot 0174+\cdot 0009h-\cdot 00002h^2+\cdot 000004h^3-\ldots=0.$$

Using Whittaker's formula we get the lowest root of this equation in the form

$$h_0=19\cdot 3423+11\cdot 6242+4\cdot 4123+2\cdot 7213+1\cdot 0215+\ldots.$$

Taking sufficient number of terms and retaining only to the second place of decimals, we get approximately

$$h_0=39\cdot 51.$$

Therefore,

$$\theta_1^2=59\cdot 71$$

or

$$\theta_1=\pm 7\cdot 72.$$

Again to find out the next higher root, we get the following equation in h

$$\cdot 0032-\cdot 00012h+\cdot 000016h^2-\cdot 000003h^3+\ldots=0.$$

By the help of Whittaker's formula, we obtain the lowest root of this equation in the form

$$h'_0=26\cdot 6732+17\cdot 3213+8\cdot 4221+4\cdot 1245+1\cdot 9234+\ldots.$$

Taking sufficient number of terms and retaining only to the second place of decimals, we get approximately

$$h'_0=59\cdot 28.$$

Therefore,

$$\theta_2^2=118\cdot 99,$$

or

$$\theta_2=\pm 10\cdot 90.$$

Before I conclude I wish to thank Prof. S. K. Banerjee for having drawn my attention to the problem discussed in this paper and for the help which I have received from him.

Potent divisors of the characteristic matrix of a minimum simple square ante-slope.*

By

Nripendranath Ghosh

[*Read February 29th, 1920*]

The most general simple square ante-slopes of order m are of two types, the square matrix (1) representing the first type and the square matrix (2) representing the second type.

$$(1)\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ 0 & a_{22} & a_{23} & \dots & a_{2m} \\ 0 & 0 & a_{33} & \dots & a_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{mm} \end{vmatrix} \quad (2)\quad \begin{vmatrix} a_{mm} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{3m} & \dots & a_{33} & 0 & 0 \\ a_{2m} & \dots & a_{23} & a_{22} & 0 \\ a_{1m} & \dots & a_{13} & a_{12} & a_{11} \end{vmatrix}.$$

Since either of the two types can be converted into the other by a symmetric derangement, it will be sufficient for our purpose to consider simple square ante-slopes of the first type. When minimum such a slope contains not more than one non-zero element in each horizontal row and not more than one non-zero element in each vertical row.

Let $A=[a]_m^m$ be a minimum simple square ante-slope of the first type. We shall find the potent divisors of its characteristic matrix or simply the characteristic potent divisors of A.

It will be proved that A can always be converted by a symmetric derangement into a compartite matrix in standard form. The successive parts of the compartite matrix can be found by grouping the diagonal elements of A in the way which will now be described.

First, put aside every non-zero element of the parametric diagonal. Let there be u such elements forming the group

$$a_{h_1h_1},\ a_{h_2h_2},\ \dots\ a_{h_uh_u}. \qquad \dots\ (H)$$

where

$$[h_1h_2\dots h_u] \qquad \dots\ (h)$$

is a corranged minor of $[1\ 2\dots m]$.

* Communicated by Prof. C. E. Cullis, M.A., Ph.D.

Next, put aside every zero-element of the parametric diagonal, which is such that neither the horizontal row nor the vertical row through it contains a non-zero element. Let there be r such elements forming the group

$$a_{k_1k_1}, a_{k_2k_2}, \cdots a_{k_rk_r}. \quad \ldots \text{(K)}$$

where $[k_1k_2\ldots k_r]$ ... (k)

is a corranged minor of $[1\ 2\ldots m]$.

Now consider the remaining zero elements of the parametric diagonal each of which is such that either the horizontal row or the vertical row through it contains a non-zero element. Let the first of these elements be $a_{p_1p_1}$. Then there is a non-zero element in the p_1th horizontal row of A. Let this be $a_{p_1p_2}$, where $p_2>p_1$. Draw straight lines joining the zero element $a_{p_1p_1}$ to $a_{p_1p_2}$ and $a_{p_1p_2}$ to the zero element $a_{p_2p_2}$. If there is any non-zero element in the p_1th horizontal row of A, let this be $a_{p_2p_3}$, where $p_3>p_2$. Draw straight lines joining the zero element $a_{p_3p_2}$ to $a_{p_2p_3}$ and $a_{p_2p_3}$ to the zero element $a_{p_3p_3}$. If there is any non-zero element in the p_3th horizontal row of A proceed as before, otherwise stop. When this process has been continued as long as possible, let the zero elements of the parametric diagonal which have been used be i in number forming the group

$$a_{p_1p_1}, a_{p_2p_2}, \cdots a_{p_ip_i}. \quad \ldots \text{(P)}$$

where $[p_1p_2\ldots p_i]$ ... (p)

is a corranged minor of $[1\ 2\ldots m]$ having no element in common with either of the sequence (h) and (k).

If there are any remaining elements of the parametric diagonal not included in the groups (H), (K), (P), let the first of these be the zero element $a_{q_1q_1}$. Then there is a non-zero element in the q_1th horizontal row of A. Let this be $a_{q_1q_2}$, when $q_2>q_1$. Draw straight lines joining the zero element $a_{q_1q_1}$ to $a_{q_1q_2}$, and $a_{q_1q_2}$ to the zero element $a_{q_2q_2}$. If there is any non-zero element in the q_2th horizontal row of A, let this be $a_{q_2q_3}$, where $q_3>q_2$. Draw straight lines joining the zero element $a_{q_2q_2}$ to $a_{q_2q_3}$, and $a_{q_1q_3}$ to the zero element $a_{q_3q_3}$. If there is any non-zero element in the q_3th horizontal row of A proceed

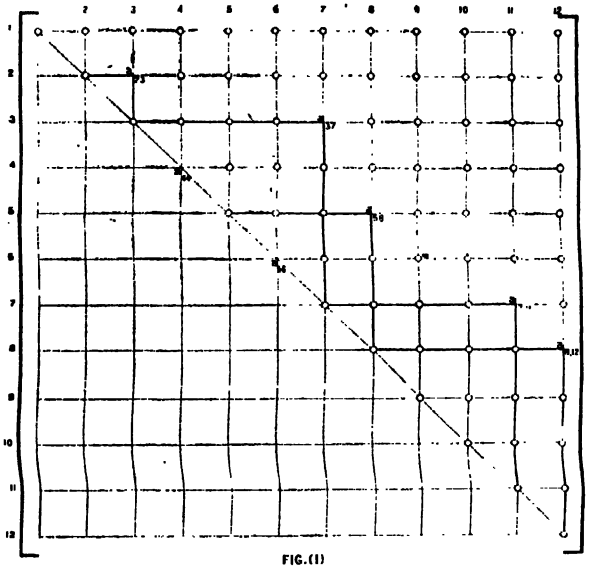


FIG.(1)

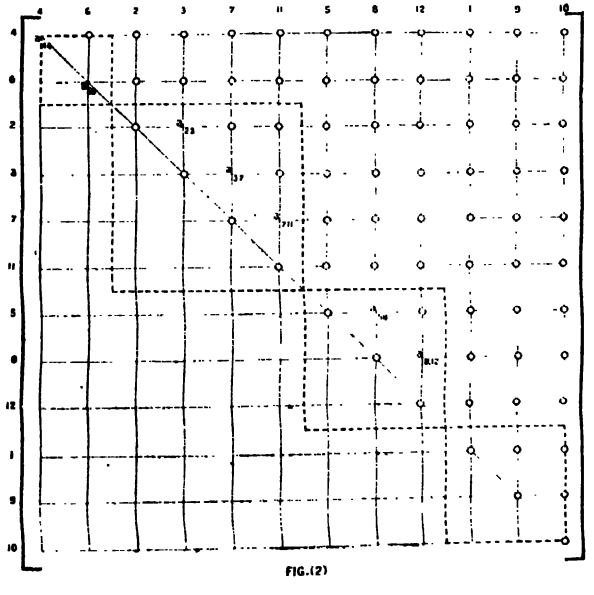


FIG.(2)

as before, otherwise stop. When this process has been continued as long as possible, let the zero elements of the parametric diagonal which have been used be j in number forming the group

$$a_{q_1q_1},\ a_{q_2q_2}\ \dots\ a_{q_jq_j}. \qquad \dots\ (Q)$$

where

$$[\,q_1q_2\dots q_j\,] \qquad \dots\ (q)$$

is a corranged minor of $[1\ 2\dots m]$ having no element in common with any one of the sequences (h), (k), (p).

In this way we can divide the elements of the parametric diagonal into the groups (h), (k), and a certain number of groups (P), (Q), (R),... all formed in succession in the same way as the group (P).

Then the symmetric minors

$$H_n=\begin{vmatrix} h_1 & h_2\dots h_n \\ & a \\ h_1 & h_2\dots h_n \end{vmatrix},\quad K_n=\begin{vmatrix} k_1 & k_2\dots k_r \\ & a \\ k_1 & k_2\dots k_r \end{vmatrix},\quad P_n=\begin{vmatrix} p_1 & p_2\dots p_r \\ & a \\ p_1 & p_2\dots p_r \end{vmatrix}\dots,$$

are mutually complementary corranged parts of A. Thus A can be converted by a symmetric derangement into a compartite matrix in standard form whose successive parts are

$$H_n, P_n, Q_n, R_n \dots K_n.$$

In the same way the characteristic matrix $A(x)$ of A can be converted into a compartite matrix in standard form whose successive parts are the characteristic matrices $H_n(x)$, $P_n(x)$, $Q_n(x)\dots K_n(x)$ of H_n, P_n, $Q_n \dots K_n$ respectively; and the potent divisors of $A(x)$ are the potent divisors of these successive parts. Thus A (x) has

n potent divisors $x-a_{h_1h_1}$, $x-a_{h_2h_2}$, $\dots x-a_{h_nh_n}$ of $H_n(x)$,

n potent divisors each being x ... of $K_n(x)$,

the single potent divisor x^r ... of $P_n(x)$,

the single potent divisor x^s ... of $Q_n(x)$,

and so on.

Consider, for example, the minimum simple square ante-slope of the first type $|a|^{12}_{12}$ (shown in Fig. 1) in which the non-zero elements are

$$a_{23},\ a_{37},\ a_{44},\ a_{58},\ a_{66},\ a_{7,11},\ a_{8,12}.$$

In this case the group H is formed by the non-zero parametric elements

$$a_{44},\ a_{66}\dots;$$

the group K consists of the zero elements

$$a_{11}, a_{99}, a_{10,10} ;$$

the group P consists of the zero elements

$$a_{22}, a_{33}, a_{77}, a_{11,11} ;$$

and the group Q consists of the zero elements

$$a_{55}, a_{88}, a_{12,12} ;$$

so that the symmetric minors are

$$H_n = \begin{bmatrix} a_{14} & 0 \\ 0 & a_{66} \end{bmatrix}, \quad K_n = \begin{bmatrix} a_{1,9} & a_{1,10} \\ a_{9,1} & a_{9,10} \end{bmatrix}, \quad P_n = \begin{bmatrix} 0 & a_{23} & 0 & 0 \\ 0 & 0 & a_{37} & 0 \\ 0 & 0 & 0 & a_{7,11} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_n = \begin{bmatrix} 0 & a_{58} & 0 \\ 0 & 0 & a_{8,12} \\ 0 & 0 & 0 \end{bmatrix}.$$

Fig. 2 shows the symmetric derangement of Fig. 1. It is a compartite matrix, the 4 successive parts being shown by dotted lines.

The characteristic potent divisors of the first part H_n are 2 in number,

$$x - a_{44}, x - a_{66} ;$$

the single characteristic potent divisor of the second part P_n is

$$x^4 ;$$

the single characteristic potent divisor of the third part Q_n is

$$x^3 ;$$

the characteristic potent divisors of the last part, the zero matrix K_n, are three in number, each being

$$x ;$$

Thus the characteristic potent divisors of $[a]_{12}^{12}$ are

$$x - a_{44}, x - a_{66}, x^4, x^3, x, x, x.$$

Liquid motion inside certain rotating curvilinear rectangles

By

Nripendranath Sen

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1. In two important memoirs Ferrers[1] and Filon[2] have discussed the problems of the liquid motion in rotating vessels and the analogous torsion problem for boundaries determined by confocal ellipses and hyperbolas. In the present paper I have shown that it is possible to obtain a solution of this problem in its most general aspect and that the different cases discussed by these two writers are deducible as particular cases of the general problem here studied. By making use of the conjugate transformation

$$x+iy=c\begin{matrix}\cosh\\ \sinh\end{matrix}(\xi+i\eta).$$

a solution is obtained for the liquid motion in a vessel bounded by $\xi=\alpha_1,\ \alpha_2$; $\eta=\beta_1,\ \beta_2$. It is then shown that by giving particular values to $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ solutions can be obtained for a number of interesting forms of boundaries including those studied by Ferrers and Filon.

I am thankful to Dr D. N. Mallik and Dr. S. K. Banerjee for their interest in the preparation of the paper.

2. For convenience, the liquid is supposed to be contained in a cylinder of unit length and confined between two smooth parallel planes at right angles to the axis of the cylinder. The cylinder is also supposed to be rotating about the centre of the confocal system. The corresponding result for rotation about any other point can be written down very easily (see *Quart. Jour.*, Vol. 17, p. 234, 1881).

[1] N. M. Ferrers, "Rotating arcs of confocal ellipse and hyperbola," *Quart. Jour.*, Vol. 17, (227-44), 1881.

[2] L. N. G. Filon, "On the resistance to torsion of certain forms of shafting with special reference to the effect of key ways." *Phil. Trans.* 193A, 309-52, 1900.

Let us consider the case of the liquid motion inside the rotating rectangle formed by two ellipses a_1, a_2 and two hyperbolas β_1, β_2; and let ω be the angular velocity of rotation.

CASE I.

3. Let $x+iy=C\cosh(\xi+i\eta)$.

Then, $$\psi=\frac{\omega c^2}{4}[\cosh 2\xi+\cos 2\eta]. \quad \ldots (1)$$

On the boundaries

(1) $\xi=a_1$, $\beta_1\leqq\eta\leqq\beta_2$.

(2) $\xi=a_2$, $\beta_1\leqq\eta\leqq\beta_2$.

(3) $\eta=\beta_1$, $a_1\leqq\xi\leqq a_2$.

(4) $\eta=\beta_2$, $a_1\leqq\xi\leqq a_2$

Assume
$$\psi=\frac{\omega c^2}{4}\Bigg[\frac{\cosh 2(\xi-\gamma)\cos 2\eta}{\cosh 2\epsilon_1}+\frac{\cosh 2\xi\cos 2(\eta-\delta)}{\cos 2\epsilon_2}$$
$$+\sum_{m=0}^{m=\infty}\Bigg\{P_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_1)$$
$$+Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_2)\Bigg\}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$+\sum_{m=0}^{m=\infty}\Bigg\{R_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)$$
$$+S_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\Bigg\}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\Bigg] \quad (2)$$

where,
$$2\epsilon_1=a_2-a_1,\quad 2\epsilon_2=\beta_2-\beta_1.$$
$$2\gamma=a_2+a_1,\quad 2\delta=\beta_2+\beta_1. \quad \ldots (3)$$

To find P_m, Q_m, R_m, S_m.

Putting $\xi=a_1$ in (1) and (2) and equating, we have,

$$\cosh 2a_1+\cos 2\eta=\cos 2\eta+\frac{\cosh 2a_1}{\cos 2\epsilon_2}\cos 2(\eta-\delta)$$
$$+\sum Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(-2\epsilon_1)\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta),$$

where $\beta_1\leqq\eta\leqq\beta_2$.

or,

$$\cosh 2\alpha_1 \left\{ -1 + \frac{\cos 2(\eta-\delta)}{\cos 2\epsilon_2} \right\}$$

$$= \Sigma Q_m \sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2} \cos \frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta),$$

where $\beta_1 \leqq \eta \leqq \beta_2$.

Multiplying both sides by $\cos \frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)d\eta$ and integrating between the limits β_1 and β_2, we have,

$$\frac{\pi}{2} Q_m \sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2}$$

$$= (-)^m 2 \cosh 2\alpha_1 \left[\frac{2m+1}{(2m+1)^2 - \frac{16\epsilon_2^2}{\pi^2}} - \frac{1}{2m+1} \right]$$

or,

$$Q_m = \frac{(-)^m 64 \epsilon_2^2 \cosh 2\alpha_1}{\pi(2m+1)\{(2m+1)^2\pi^2 - 16\epsilon_2^2\} \sinh \pi(2m+1)\epsilon_1} \quad \ldots \quad (4)$$

Putting $\xi = \alpha_2$ and proceeding exactly as above, we have,

$$P = \frac{(-)^{m+1} 64 \epsilon_2^2 \cosh 2\alpha_2}{\pi(2m+1)\{(2m+1)^2\pi^2 - 16\epsilon_2^2\} \sinh \pi(2m+1)\epsilon_1} \quad \ldots \quad (5)$$

Similarly, putting $\eta = \beta_1$ in (1) and (2), equating, multiplying both sides by $\cos \frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\,d\xi$ and integrating between the limits α_1 and α_2, we have,

$$S_m = \frac{(-)^{m+1} 64 \epsilon_1^2 \cos 2\beta_1}{\pi(2m+1)\{(2m+1)^2\pi^2 + 16\epsilon_1^2\} \sinh(2m+1)\frac{\pi\epsilon_2}{\epsilon_1}} \quad \ldots \quad (6)$$

Similarly, putting $\eta = \beta_2$ and proceeding exactly as above, we have,

$$R_m = \frac{(-)^m 64 \epsilon_1^2 \cos 2\beta_2}{\pi(2m+1)\{(2m+1)^2\pi^2 + 16\epsilon_1^2\} \sinh \frac{(2m+1)\pi\epsilon_2}{\epsilon_1}} \quad \ldots \quad (7)$$

Thus from (2), (3), (4), (5), (6) and (7), ψ is completly determined

Observing that $\phi+i\psi=f(\xi+i\eta)$, we can easily deduce the expression for ϕ from that for ψ.

Thus,

$$\phi=-\frac{\omega c^2}{4}\left[\frac{\sinh 2(\xi-\gamma)\sin 2\eta}{\cosh 2\epsilon_1}+\frac{\sinh 2\xi \sin 2(\eta-\delta)}{\cos 2\epsilon_2}\right.$$

$$+\sum_{m=0}^{m=\infty}\left\{P_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)+Q_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\}$$

$$\times\sin\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$

$$-\sum_{m=0}^{m=\infty}\left\{R_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)+S_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}$$

$$\left.\times\sin\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right] \quad \ldots \quad (8)$$

The expression for the Kinetic Energy T is easily seen to be

$$\frac{32T}{\rho\omega^2c^4}=\frac{1}{2}\tan 2\epsilon_2\,(\sinh 4\alpha_2-\sinh 4\alpha_1)$$

$$-\frac{1}{2}\tanh 2\epsilon_1\,(\sin 4\beta_2-\sin 4\beta_1)$$

$$+2\epsilon_2\left\{2\tanh 2\epsilon_2+(\sinh 2\alpha_2-\sinh 2\alpha_1)\frac{\cos 2\delta}{\cos 2\epsilon_2}\right\}$$

$$-2\epsilon_1\left\{2\tan 2\epsilon_2+(\sin 2\beta_2-\sin 2\beta_1)\frac{\cosh 2\gamma}{\cosh 2\epsilon_1}\right\}$$

$$\frac{32}{\pi}\sum_{m=0}^{m=\infty}\frac{}{(2m+1)(M^2-4)^2}\{M(\sinh 4\alpha_2-\sinh 4\alpha_1)$$

$$-2(\cosh 4\alpha_2+\cosh 4\alpha_1+2)\coth\theta+8\cosh 2\alpha_2\cosh 2\alpha_1\operatorname{cosech}\theta\}$$

$$-\frac{32}{\pi}\sum_{m=0}^{m=\infty}\frac{1}{(2m+1)(N^2+4)^2}\{N(\sin 4\beta_2-\sin 4\beta_1)$$

$$-2(\cos 4\beta_2+\cos 4\beta_1+2)\coth\theta'+8\cos 2\beta_2\cos 2\beta_1\operatorname{cosech}\theta'\}$$

$$-\frac{64}{\pi}(\cosh 2\alpha_2+\cosh 2\alpha_1)(\cos 2\beta_2+\cos 2\beta_1)$$

$$\times\sum_{m=0}^{m=\infty}\frac{1}{2m+1}\left\{\frac{\tanh\frac{\theta}{2}}{(M^2-4)^2}+\frac{\tanh\frac{\theta'}{2}}{(N^2+4)^2}\right\}, \quad \ldots \quad (9)$$

where, $M=\frac{(2m+1)\pi}{2\epsilon_2}$; $N=\frac{(2m+1)\pi}{2\epsilon_1}$;

$$\theta=\frac{(2n+1)\pi\epsilon_1}{\ldots} \; ; \; \theta'=\frac{(2n+1)\pi\epsilon_2}{\ldots} . \qquad \ldots \quad (10)$$

The following particular cases are interesting :—

(1) Liquid motion inside a rotating rectangle formed by two ellipses α_2, α_1 and the hyperbola β.

The expressions for ϕ and ψ for this case can be easily deduced by putting $-\beta_1=\beta_2=\beta$ (say) and noting that $2\epsilon_2=\beta_2-\beta_1=2\beta$ and $2\delta=\beta_2+\beta_1=0$.

(2) Liquid motion inside a rotating rectangle formed by two hyperbolas β_2, β_1 and the ellipse α.

The results for this case can be obtained by writing $\alpha_1=-\alpha_2=-\alpha$ and noting that

$$\epsilon_1=\frac{\alpha_2-\alpha_1}{2}=\alpha, \; \gamma=\frac{\alpha_2+\alpha_1}{2}=0.$$

(3) Liquid motion inside the figure formed by one ellipse α and one branch of a hyperbola β.

For this case put $\alpha_1=-\alpha_2=-\alpha$. $\beta_1=-\beta_2=-\beta$. $\epsilon_1=\alpha$, $\epsilon_2=\beta$, $\gamma=0$, $\delta=0$. This case has been studied by Filon and Ferrers. The expressions obtainted by the above substitutions are easily seen to be identical with those obtained by the latter.

(4) Liquid motion inside the curvilinear rectangle formed by two concentric circles (α_1, α_2) and two bounding radii (β, $-\beta$)

This can be very easily deduced by diminishing c and increasing α_1, α_2, and ξ indefinitely such that

$$c\cosh\alpha_1=c\sinh\alpha_1=a_1 \; ; \; c\cosh\xi=c\sinh\xi=r \; ;$$

$$c\cosh\alpha_2=c\sinh\alpha_2=a_2 \; ; \quad \xi-\gamma=\tfrac{1}{2}\log\frac{a_2}{ab}$$

$$2\epsilon_1=(\alpha_2-\alpha_1)=\log\frac{a_2}{a_1} .$$

Also let $\beta_1=-\beta_2=-\beta$

so that $\epsilon_2=\beta$. $\delta=0$.

Then after a little simplification we have.

$$\psi=\frac{\omega}{2}\left[\frac{r^2\cos 2\eta}{\cos 2\beta}+\sum_{m=0}^{m=\infty}M\left\{a_2^{\,2}\sinh\frac{(2m+2)\pi}{2\beta}\log\frac{r}{a_1}\right.\right.$$

$$\left.\left.-a_1^{\,2}\sinh\frac{(2m+1)\pi}{2\beta}\log\frac{r}{a^2}\right\}\cos\frac{(2m+1)\pi}{2\beta}\eta\right],\quad\ldots\quad(1)$$

$$\phi=-\frac{\omega}{2}\left[\frac{r^2\sin 2\eta}{\cos 2\beta}+\sum_{m=0}^{m=\infty}M\left\{a_2^{\,2}\cosh\frac{(2m+1)\pi}{2\beta}\log\frac{r}{a_1}\right.\right.$$

$$\left.\left.-a_1^{\,2}\cosh\frac{(2m+1)\pi}{2\beta}\log\frac{r}{a_2}\right\}\sin\frac{(2m+1)\pi}{2\beta}\eta\right].\quad\ldots\quad(2)$$

where

$$M=\frac{(-)^{m+1}\,64\beta^2}{\pi(2m+1)\{(2m+1)^2\pi^2-16\beta^2\}\sinh\dfrac{\pi(2m+1)\log\dfrac{a_2}{a_1}}{2\beta}}.\quad\ldots\quad(3)$$

The above results are simpler than those of Sir A. George Greenhill,[1] into which they can, however, be transformed after a little reduction.

(5) Liquid motion in a rotating elliptic quadrant.

This can be deduced very easily as follows. Let $a_1=0$, $\beta=0$, $a_2=a$, $\beta_2=\frac{\pi}{2}$. Evidently, the curvilinear rectangle takes the shape of an elliptic quadrant. But the expression for ϕ and ψ and T take the undetermined forms

$$\infty-\infty$$

which, however, can be evaluated by putting

$$\beta_2=\frac{\pi}{2}+\theta \text{ where } Lt.\ \theta=0.$$

[1] Sir A. G. Greenhill—*Mess. Math.*, Vol. IX, p. 35, 1880.

The expressions for ϕ and ψ thus found out are as follows:

$$\phi = \frac{\omega c^2}{4}\left[-\frac{4}{\pi}\{(\xi-a)\cos 2\xi\cos 2\eta-\left(\eta-\frac{\pi}{2}\right)\sinh 2\xi\sin 2\eta\right.$$

$$\left.-\frac{3}{4}\sinh 2\xi\cos 2\eta-\frac{\pi}{4}\tanh a\cosh 2\xi\sin 2\eta\right\}$$

$$+\sum_{m=1}^{m=\infty}M\{\cos 2(2m+1)(\xi-a)-\cosh 2a\cosh 2(2m+1)\xi\}$$

$$\times\cos 2(2m+1)\eta$$

$$+\sum_{m=0}^{m=\infty}N\cosh\frac{(2m+1)\pi}{a}\left(\eta-\frac{\pi}{4}\right)\cos\frac{(2m+1)\pi\xi}{a}\Big], \quad \ldots \quad (1)$$

$$\psi = \frac{\omega c^2}{4}\left[-\frac{4}{\pi}\{(\xi-a)\sinh 2\xi\sin 2\eta+\left(\eta-\frac{\pi}{2}\right)\cosh 2\xi\cos 2\eta\right.$$

$$\left.-\frac{3}{4}\sin 2\eta\cosh 2\xi+\frac{\pi}{4}\tanh a\sinh 2\xi\cos 2\eta\right\}$$

$$+\sum_{m=1}^{m=\infty}M\{\sinh 2(2m+1)(\xi-a)-\cosh 2a\sinh 2(2m+1)\xi\}$$

$$\times\sin 2(2m+1)\eta$$

$$-\sum_{m=0}^{m=\infty}N\sinh\frac{(2m+1)\pi}{a}\left(\eta-\frac{\pi}{4}\right)\sin\frac{(2m+1)\pi\xi}{a}\Big]. \quad \ldots \quad (2)$$

where $$M=\frac{1}{\pi m(2m+1)(m+1)\sinh 2(2m+1)a}. \quad \ldots \quad (3)$$

$$N=\frac{16a^2}{\pi(2m+1)\{(2m+1)^2\pi^2+4a^2\}\sinh\frac{(2m+1)\pi^2}{4a}}. \quad \ldots \quad (4)$$

Case II.

Now let $x=c\cosh\xi\sin\eta$, $y=c\sinh\xi\cos\eta$.

We have then $$\psi=\frac{\omega c^2}{4}(\cosh 2\xi-\cos 2\eta), \quad \ldots \quad (1)$$

and on the boundaries (1) $\xi=a_1,\quad \beta_1\leqq\eta\leqq\beta_2,$

(2) $\xi=a_2,\quad \beta_1\leqq\eta\leqq\beta_2,$

(3) $\eta=\beta_1,\quad a_1\leqq\xi\leqq a_2,$

(4) $\eta=\beta_2,\quad a_1\leqq\xi\leqq a_2.$

Assume
$$\psi=\frac{\omega c^2}{4}\left[\frac{\cosh 2\xi\cos 2(\eta-\delta)}{\cos 2\epsilon_2}-\frac{\cosh 2(\xi-\gamma)\cos 2\eta}{\cosh 2\epsilon_1}\right.$$
$$+\sum_{m=0}^{m=\infty}\left\{P_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_1)\right.$$
$$\left.+Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_2)\right\}\times\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$+\sum_{m=0}^{m=\infty}\left\{R_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)\right.$$
$$\left.\left.+S_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}\times\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right],\quad\ldots\quad(2)$$

where
$$2\epsilon_1=a_2-a_1,\quad 2\gamma=a_2+a_1.\qquad\ldots\quad(3)$$
$$2\epsilon_2=\beta_2-\beta_1,\quad 2\delta=\beta_2+\beta_1.$$

Proceeding exactly as in Case I, we have,

$$P_m=\frac{(-)^{m+1}64\epsilon_2^2\cosh 2a_2}{\pi(2m+1)\{(2m+1)^2\pi^2-16\epsilon_2^2\}\sinh\frac{(2m+1)\pi\epsilon_1}{\epsilon_2}},$$

$$Q_m=-\frac{P_m}{\cosh 2a_2}\cosh 2a_1,$$

$$R_m=\frac{(-)^{m+1}64\epsilon_1^2\cos 2\beta_2}{\pi(2m+1)\{(2m+1)^2\pi^2+16\epsilon_1^2\}\sinh\frac{(2m+1)\pi\epsilon_2}{\epsilon_1}},$$

$$S_m=-\frac{R_m}{\cos 2\beta_2}\cos 2\beta_1.\qquad\ldots\quad(4)$$

Thus, ψ is completely determined from (2), (3) and (4).

Since $\phi+i\psi=f(\xi+i\eta)$, ϕ can be deduced from (2) very easily.

Thus
$$\phi=-\frac{\omega c^2}{4}\left[\frac{\sinh 2\xi \sin 2(\eta-\delta)}{\cos 2\epsilon_2}-\frac{\sinh 2(\xi-\gamma)\sin 2\eta}{\cosh 2\epsilon_1}\right.$$
$$+\sum_{m=0}^{m=\infty}\{P_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_1)$$
$$+Q_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_2)\}\sin\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$-\sum_{m=0}^{m=\infty}\{R_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)$$
$$\left.+S_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\}\sin\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right]. \quad \ldots \quad (5)$$

The expression for the kinetic energy can be worked out as before.

The following particular cases can be deduced from the above expressions :—

(1) Liquid motion inside two ellipses and two branches of the same hyperbola β.

For this, put $2\epsilon_2=\beta$. $\delta=0$ in the above expressions for ϕ and ψ.

(2) Liquid motion inside two hyperbolas and one ellipse (α).

For this put $2\epsilon_1=2\alpha$, $\gamma=0$ in (2) and (5).

(3) Liquid motion inside one ellipse (α) two branches of the same hyperbola (β).

For this, put $a_1=-a_2=-\alpha$

$\beta_1=-\beta_2=-\beta$ in (2) and (5).

So that $\epsilon_1=\alpha$, $\epsilon_2=\beta$, $\gamma=0$. $\delta=0$.

The expressions for ϕ and ψ are thus obtained. The results thus deduced are found to be identical with those of Ferrers.[1]

[1] Ferrers. *ibid.*

(*d*) Liquid motion inside a rectangular box $(2a, 2b)$ rotating about the origin with its centre at (x_1, y_1).

This is deduced very easily from the above expressions by increasing c and diminishing α, ξ, β, η, indefinitely, such that

$$c\eta = x, \quad c\xi = y,$$

$$c\epsilon_2 = a, \quad c\epsilon_1 = b. \qquad \ldots \quad (1)$$

$$c\delta = x_1, \quad c\gamma = y_1,$$

we thus get,

$$\phi = \omega[x_1 y - y_1 x$$

$$- \sum_{m=0}^{m=\infty} P_m \sinh \pi \frac{(2m+1)(y-y_1)}{2a} \sin \frac{(2m+1)\pi}{2a}(x-x_1)$$

$$+ \sum_{m=0}^{m=\infty} Q_m \sinh \frac{(2m+1)\pi}{2b}(x-x_1) \sin \frac{(2m+1)\pi}{2b}(y-y_1)], \qquad \ldots \quad (2)$$

$$\psi = \omega[xx_1 + yy_1 - C$$

$$+ \sum_{m=0}^{m=\infty} P_m \cosh \frac{\pi(2m+1)(y-y_1)}{2a} \cos \frac{(2m+1)\pi}{2a}(x-x_1)$$

$$+ \sum_{m=0}^{m=\infty} Q_m \cosh \frac{\pi(2m+1)(x-x_1)}{2b} \cos \frac{(2m+1)\pi}{2b}(y-y_1)], \qquad \ldots \quad (3)$$

where

$$P_m = \frac{(-)^{m+1} 16a}{\pi^3 (2m+1)^3 \cosh \frac{(2m+1)\pi b}{2a}}, \qquad \ldots \quad (4)$$

$$Q_m = \frac{(-)^{m+1} 16b^3}{\pi^3 (2m+1)^3 \cosh \frac{(2m+1)\pi a}{2b}}.$$

C = a constant.

Putting $x_1 = 0$, $y_1 = 0$, we can at once deduce the well-known[1] result for the liquid motion inside a rectangular box rotating about its centre.

[1] Ferrers.—*Quart. Jour.*, Vol. 15, p. 183, 1877; Vol. 17, p. 231, 1879.

The following two alternative solutions can also be easily obtained (for ψ and ϕ in Case I):—

$$\psi=\frac{\omega c^2}{4}\left[\cos 2\delta\cos 2\epsilon_2-\sin 2\delta\sin 2\epsilon_1\frac{\theta}{\epsilon_2}+\frac{\cosh 2\xi\cos 2\theta}{\cos 2\epsilon_2}\right.$$

$$+\frac{16}{\epsilon_2}\sin 2\epsilon_2\sin 2\delta\sum_{m=1}^{m=\infty}\frac{(-)^m}{2m\left(\frac{m^2\pi^2}{\epsilon_2^2}-4\right)}\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\cosh\frac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi}{\epsilon_2}\theta.$$

$$+16\sum_{m=0}^{m=\infty}\frac{(-)^{m+1}}{(2m+1)\left\{\frac{(2m+2)^2\pi^2}{4\epsilon_2^2}-4\right\}}\cos\frac{(2m+1)\pi\theta}{2\epsilon_2}$$

$$\times\left\{(\cosh 2\gamma\cosh 2\epsilon_1-\cos 2\delta\cos 2\epsilon_2)\frac{\cosh\frac{(2m+1)\pi\phi}{2\epsilon_2}}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\right.$$

$$\left.\left.+\sinh 2\gamma\sinh 2\epsilon_1\frac{\sinh\frac{(2m+1)\pi\phi}{2\epsilon_2}}{\sinh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\right\}\right],$$

$$\psi=\frac{\omega c^2}{4}\left[\cosh 2\gamma\cos 2\epsilon_1+\sinh 2\gamma\sinh 2\epsilon_1\frac{\phi}{\epsilon_1}+\frac{\cosh 2\phi\cos 2(\theta+\delta)}{\cosh 2\epsilon_1}\right.$$

$$+\frac{16}{\epsilon_1}\sinh 2\gamma\sinh 2\epsilon_1\sum_{m=1}^{m=\infty}\frac{(-)^m}{2m\left\{\frac{m^2\pi^2}{\epsilon_1^2}+4\right\}}\frac{\cosh\frac{m\pi\theta}{\epsilon_1}}{\cosh\frac{m\pi\epsilon_2}{\epsilon_1}}\sin\frac{m\pi\phi}{\epsilon_1}$$

$$+16\sum_{m=0}^{m=\infty}\frac{(-)^{m+1}}{(2m+1)\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4\right\}}\cos\frac{(2m+1)\pi\phi}{2\epsilon_1}$$

$$\times\left\{(\cosh 2\gamma\cosh 2\epsilon_1-\cos 2\delta\cos 2\epsilon_2)\frac{\cosh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\right.$$

$$\left.\left.+\sin 2\delta\sin 2\epsilon_2\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\right\}\right],$$

where $\phi=\xi-\gamma;\ \theta=\eta-\delta.$

The corresponding expressions for ϕ can be easily written down.

Similar alternative expressions for ϕ and ψ can also be obtained for Case II and from these Filon's solutions may be deduced by putting

$$\gamma=0,\ \epsilon_1=a.$$

The proof of these expressions for ϕ and ψ depends on the identity

$$\frac{\pi}{16}\frac{\cosh 2\phi \sin 2\theta}{\cosh 2\epsilon_1 \sin 2\epsilon_2}-\frac{\pi\theta}{16\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m+1}$$

$$\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\left\{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4\right\}\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi$$

$$+\sum_{m=1}^{m=\infty}\left(\frac{(-)^m}{2m\frac{m^2\pi^2}{\epsilon_2^2}-4}\right)\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\cosh\frac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2}=0,$$

and two others.

This identity may be proved by starting from the expression

$$u=-\frac{\pi\theta}{4\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m+1}\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi$$

$$+\sum_{m=1}^{m=\infty}\frac{(-)^{m-1}}{2m}\frac{\cosh\frac{m\pi}{\epsilon_2}\phi}{\cosh\frac{m\pi\epsilon_1}{\epsilon_2}}\sin\frac{m\pi\theta}{\epsilon_2},$$

and noting that the potential function u vanishes on the boundaries defined by $(\xi=a_1, a_2;\ \eta=\beta_1, \beta_2)$ and therefore at every point inside the region enclosed by these boundaries. Now since the two operators $\frac{\partial^2}{\partial\theta^2}+4$ and $-\frac{\partial^2}{\partial\phi^2}+4$ are identical, if we perform the operations $\left(\frac{\partial^2}{\partial\theta^2}+4\right)^{-1}$ and $\left(-\frac{\partial^2}{\partial\phi^2}+4\right)^{-1}$ on both sides of the above expression with

n put equal to zero, we easily obtain

$$0=-\frac{\pi\theta}{16\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{(2m+1)}\left\{\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4}\right\}\sinh\frac{(2m+1)\pi\epsilon}{2\epsilon_1}\cos\frac{(2m+1)\pi}{2\epsilon}\phi$$

$$\sum_{m=1}^{m=\infty}\frac{(-)^m}{2m}\cdot\left\{\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\frac{m^2\pi^2}{\epsilon_2^2}-4}\right\}\cosh\frac{m\pi\epsilon_1}{\epsilon_2}\sin\frac{m\pi\theta}{\epsilon_2}+f(\phi)\sin 2\theta$$

$$=-\frac{\pi\theta}{16\epsilon_2}+\sum_{m=0}^{m=\infty}\frac{(-)^m}{(2m+1)}\left\{\frac{\sinh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4}\right\}\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}\cos\frac{(2m+1)\pi\phi}{2\epsilon_1}$$

$$+\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m}\cdot\left(\frac{\cosh\frac{m\pi\phi}{\epsilon_2}}{\frac{m^2\pi^2}{\epsilon_2^2}-4}\right)\cosh\frac{m\pi\epsilon_1}{\epsilon_2}\sin\frac{m\pi\theta}{\epsilon_2}+\mathrm{F}(\theta)\cosh 2\phi.$$

where $$-\epsilon_1<\phi<\epsilon_1,\ -\epsilon_2<\theta<\epsilon_2.$$

Hence $f(\phi)=c\cosh 2\phi$, $\mathrm{F}(\theta)=c\sin 2\theta$ and by putting $\phi=\epsilon_1$, $\theta=\epsilon_2$, we obtain $c=\frac{\pi}{16\cosh 2\epsilon_1\sin 2\epsilon_2}$ which proves the identity. A similar identity is obtained by starting with an expression for n in which θ, ϕ and ϵ_1, ϵ_2 have been interchanged. If on the other hand we start with the expression,

$$n=\sum_{m=0}^{m=\infty}\frac{(-)^m}{2m+1}\left\{\frac{\cosh\frac{(m+1)\pi\phi}{2\epsilon_2}}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi\theta}{2\epsilon_2}\right.$$

$$\left.+\frac{\cosh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}\phi\right\},$$

we get,

$$\frac{\pi}{16}\left[1-\frac{\cosh 2\phi \cos 2\theta}{\cosh 2\epsilon_1 \cos 2\epsilon_2}\right] = \sum_{m=0}^{m=\infty} \left\{ \frac{(-)^{m+1}}{(2m+1)\left(\frac{(2m+1)^2\pi^2}{4\epsilon_2^2}-4\right)} \right.$$

$$\left. \frac{\cosh\frac{(2m+1)\pi\phi}{2\epsilon_2}}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}} \cos\frac{(2m+1)\pi\theta}{2\epsilon_2} \right.$$

$$+\frac{}{(2m+1)}\left\{\frac{(-)^m}{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}+4}\right\}+\frac{\cosh\frac{(2m+1)\pi\theta}{2\epsilon_1}}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi\phi}{2\epsilon_1}\Bigg\}$$

(*vide* Ferrers. l.c.).

On the Horpolhode[1]

By

Satyendranath Basu

M. de Sparre has shown that the horpolhode of Poinsot contains no point of inflexion; his proof was based upon the properties of elliptic functions. Various other proofs of the theorem have since been given by Mannheim, Saint-Germain,[2] Routh,[3] Le-Cornu[4] and others.

The following simple proof of the theorem is based solely on the dynamical considerations of the Poinsot motion; hence it might appear interesting and instructive.

Let A, B, C be the three principal moments of inertia of a body fixed at a point O; let OI and OI′ be the two instantaneous axes, at time t and $t+dt$. I, I′, etc., on the invariable plane trace out the horpolhode, length of OI, OI′, etc., being proportional to the resulting angular velocity at all times.

At a point of inflexion of the horpolhode, there will be a stationary tangent, *i.e.*, the total change of the vector II′ will be then along its own length.

Remarking that II′ has components proportional to $\dot{\omega}_1 dt$, $\dot{\omega}_2 dt$, $\dot{\omega}_3 dt$, along the three moving principal axes, the condition reduces to

$$(1)\quad \frac{\ddot{\omega}_1 - \dot{\omega}_2\omega_3 + \dot{\omega}_3\omega_2}{\dot{\omega}_1} = \frac{\ddot{\omega}_2 - \dot{\omega}_3\omega_1 + \dot{\omega}_1\omega_3}{\dot{\omega}_2} = \frac{\ddot{\omega}_3 - \dot{\omega}_1\omega_2 + \dot{\omega}_2\omega_1}{\dot{\omega}_3}$$

$$= \lambda \text{ say.}$$

[1] An exactly similar proof of the theorem discussed in this paper has been given by Prof. W. Van der Woude in a paper entitled "Over de Herpolõide Van Poinsot," published in *Nieuw Archief voor Wiskunde*, Tweede Reeks, Deel XIII, Eerste Stuk, pp. 94, 95 (1919). It appears however from an enquiry that this journal was not received in the library of the Society before the beginning of November, 1919.—S.K.B.

[2] Compte Rendu—1885.

[3] Advanced Dynamics.

[4] Bulletin de Soc. Math. de France—1906.

We have from Euler's equations

$$A\dot{\omega}_1=(B-C)\omega_2\omega_3,$$

$$B\dot{\omega}_2=(C-A)\omega_3\omega_1,$$

$$C\dot{\omega}_3=(A-B)\omega_1\omega_2\ ;$$

from which

$$A\ddot{\omega}_1=(B-C)\,(\dot{\omega}_2\omega_3+\dot{\omega}_3\omega_2),$$

$$B\ddot{\omega}_2=(C-A)(\dot{\omega}_3\omega_1+\omega_1\dot{\omega}_3),$$

$$C\ddot{\omega}_3=(A-B)\,(\dot{\omega}_1\omega_2+\dot{\omega}_2\omega_1).$$

Substituting in (1) we have

$$\frac{\dot{\omega}_3}{\omega_3}(A+B-C)-\frac{\dot{\omega}_2}{\omega_2}(C+A-B)=\lambda(B-C)$$

and two similar equations.

So that

$$\frac{\dot{\omega}_1}{\omega_1}(B-C)(B+C-A)+\frac{\dot{\omega}_2}{\omega_2}(C-A)(C+A-B)+\frac{\dot{\omega}_3}{\omega_3}(A-B)$$
$$(A+B-C)=0.$$

But remembering that

$$A\omega_1\dot{\omega}_1+B\omega_2\dot{\omega}_2+C\omega_3\dot{\omega}_3=0.$$

$$A^2\omega_1\dot{\omega}_1+B^2\omega_2\dot{\omega}_2+C^2\omega_3\dot{\omega}_3=0,$$

we have

$$\frac{\omega_1\dot{\omega}_1}{BC(B-C)}=\frac{\omega_2\dot{\omega}_2}{CA(C-A)}=\frac{\omega_3\dot{\omega}_3}{AB(A-B)}$$

hence substituting, we get as a necessary condition for the point of inflexion.

$$\frac{BC(B-C)^2\,(B+C-A)}{\omega_1^2}+\frac{CA(C-A)^2(C+A-B)}{\omega_2^2}$$
$$+\frac{AB(A-B)^2(A+B-C)}{\omega_3^2}=0$$

which is obviously impossible because B+C−A, C+A−B, A+B−C, are all positive, so that the left-hand side is an essentially positive quantity.

On a Generalization of Neumann's Expansion in a Series of Bessel Functions

By

Abanibhushan Datta

1. Neumann has shown that any arbitrary function, regular at the origin, can be expanded in a series of Bessel Functions of the form:

$$f(z)=a_0J_0(z)+a_1J_1(z)+\ldots.$$

where the co-efficients a_0, a_1, a_2, ... are independent of z. In the present paper Neumann's method has been usefully employed to obtain the expansion of any arbitrary function in terms of Bessel Function in the following more general forms:

$$f(z)=\sum b_n J_{n+\alpha}(kz), \tag{I}$$

$$f(z)=\sum b_n \frac{J_{n+\alpha}(kz)}{z^n}. \tag{II}$$

where the co-efficients b_n, etc., are independent of z, and α is any positive integer or fraction as the case may be. An interesting application of the results obtained here consists in obtaining defferent forms of expansion for the same function in terms of Bessel Functions of integral or fractional orders. Some simple examples have been given towards the end of this paper. It may be remarked that a series in Bessel Functions of increasing orders is usually convergent. It may however be necessary to test the convergence in some of the particular cases given hereafter.

2. Let us first consider the expansion in the form (I).

Let $$f(z)=\sum_{n=0}^{n=\infty} b_n J_{n+\alpha}(kz),$$

where α is an integer.

Let us try to determine the co-efficients by expanding both sides of the equation as power series in z and equating co-efficients of the several powers of z.

Now $$f(z)=f(o)+zf'(o)+\frac{z^2}{2!}f''(o)+\ldots$$

and $$J_{n+a}(kz)=\left(\frac{k}{2}\right)^{n+a}\frac{z^{n+a}}{\Gamma(n+a+1)}\left[1-\left(\frac{k}{2}\right)^2\frac{z^2}{(n+a+1)}\right.$$

$$\left.+\left(\frac{k}{2}\right)^4\frac{z^4}{2!\,(n+a+1)(n+a+2)}-\ldots\right].$$

Hence $f(o)=o$, $f'(o)=o\ldots f^r(o)=o$ where $r<a$.

When $r=a$,

$$\frac{f^a(o)}{a!}=\left(\frac{k}{2}\right)^a\frac{1}{\Gamma(a+1)}\,b_0$$

$$\frac{f^{a+1}(o)}{(a+1)!}=\left(\frac{k}{2}\right)^{1+a}\frac{1}{\Gamma(a+2)}\,b_1$$

$$\frac{f^{a+2}(o)}{(a+2)!}=-\left(\frac{k}{2}\right)^{a+2}\frac{b_0}{\Gamma(a+2)}+\left(\frac{k}{2}\right)^{a+2}\frac{b_2}{\Gamma(a+3)}$$

$$\frac{f^{a+3}(o)}{(a+3)!}=-\left(\frac{k}{2}\right)^{a+3}\frac{b_1}{\Gamma(a+3)}+\left(\frac{k}{2}\right)^{a+3}\frac{b_3}{\Gamma(a+4)}$$

$$\frac{f^{a+4}(o)}{(a+4)!}=\left(\frac{k}{2}\right)^{a+4}\frac{b_0}{2!\,\Gamma(a+3)}-\left(\frac{k}{2}\right)^{a+4}\frac{b_2}{\Gamma(a+4)}$$

$$+\left(\frac{k}{2}\right)^{a+4}\frac{b_4}{\Gamma(a+5)}$$

$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

$$\frac{f^{a+2n}(o)}{(a+2n)!}=(-1)^n\left(\frac{k}{2}\right)^{a+2n}\times\left[\frac{b_0}{n!\,\Gamma(a+n+1)}\right.$$

$$\left.-\frac{b_2}{(n-1)!\,\Gamma(n+a+2)}+\ \ldots\ +(-1)^n\frac{b_{2n}}{\Gamma(2n+a+1)}\right]$$

$$\frac{f^{a+2n+1}(o)}{(a+2n+1)}=(-1)^n\left(\frac{k}{2}\right)^{2n+a+1}\left[\frac{b_1}{n!\,\Gamma(n+a+2)}\right.$$

$$\left.-\frac{b_3}{(n-1)!\,\Gamma(n+a+3)}+\ \ldots\ +(-1)^n\frac{b_{2n+1}}{\Gamma(2n+a+2)}\right].$$

From the above identities, we have

$$b_0=\left(\frac{2}{k}\right)^a f^a(o)$$

$$b_1=\left(\frac{2}{k}\right)^{1+a} f^{a+1}(o)$$

$$b_2=\left(\frac{2}{k}\right)^{a+2}\Gamma(a+3)\left[\left(\frac{k}{2}\right)^{a+2}\frac{b_0}{\Gamma(a+2)}+\frac{f^{a+2}(o)}{(a+2)!}\right]$$

$$=\left(\frac{2}{k}\right)^a\left[(a+2)f^a(o)+\left(\frac{2}{k}\right)^2 f^{a+2}(o)\right]$$

$$b_3=\left(\frac{2}{k}\right)^{a+1}\Gamma(a+4)\left[\left(\frac{k}{2}\right)^{a+3}\frac{b_1}{\Gamma(a+3)}+\frac{f^{a+3}(o)}{(a+3)!}\right]$$

$$=\left(\frac{2}{k}\right)^{a+1}\left[(a+3)f^{a+1}(o)+\left(\frac{2}{k}\right)^2 f^{a+3}(o)\right]$$

$$b_4=(-1)^3\Gamma(a+5)\left[\frac{b_0}{2!\,\Gamma(a+3)}-\frac{b_2}{\Gamma(a+4)}\right.$$

$$\left.+(-)\left(\frac{2}{k}\right)^{a+4}\frac{f^{a+4}(o)}{(a+4)!}\right]$$

$$=(-)^3\left(\frac{2}{k}\right)^a\left[f^a(o)\left\{\frac{\Gamma(a+5)}{2!\,\Gamma(a+3)}-\frac{\Gamma(a+5)}{\Gamma(a+4)}(a+2)\right\}\right.$$

$$\left.-(a+4)f^{a+2}(o)\left(\frac{2}{k}\right)^2+(-\left(\frac{2}{k}\right)^4 f^{a+4}(o)\right]$$

... ...

$$b_{2n}=(-)^{n+1}\Gamma(2n+a+1)\left[\frac{b_0}{n!\,\Gamma(n+a+1)}-\frac{b_2}{(n-1)!\,\Gamma(n+a+2)}\right.$$

$$\left.+\ \dots\ +(-1)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n+a}(o)}{(2n+a)!}\right]$$

$$=(-1)^{n+1}\left(\frac{2}{k}\right)^a\left[\Gamma(2n+a+1)f^a(o)\left\{\frac{1}{n!\,\Gamma(n+a+1)}\right.\right.$$

$$-\frac{\Gamma(a+3)}{(n-1)\Gamma(n+a+2)}-\left(\frac{\Gamma(a+5)}{2!\,\Gamma(a+3)}\right.$$

$$\left.\left.-\frac{\Gamma(a+5)}{\Gamma(a+4)}\cdot\frac{\Gamma(a+3)}{\Gamma(a+2)}\right)\frac{1}{(n-2)!\,\Gamma(n+a+3)}-\dots\right\}$$

$$-\left(\frac{2}{k}\right)^3\Gamma(2n+a+1)f^{a+3}(o)\left\{\frac{1}{(n-1)!\,\Gamma(a+n+2)}\right.$$

$$\left.-\frac{\Gamma(a+5)}{\Gamma(a+4)}\cdot\frac{1}{(n-2)!\,\Gamma(n+a+3)}-\ \cdots\ \right\}$$

$$+$$

$$+|(-)^{n-2}\left(\frac{2}{k}\right)^{2n-4+a}f^{2n+a-4}(o)\left\{\frac{\Gamma(2n+a+1)}{\Gamma(2n+a-1)2!}\right.$$

$$\left.-\frac{\Gamma(2n+a+1)}{\Gamma(2n+a)}\cdot\frac{\Gamma(2n+a-1)}{\Gamma(2n+a-2)}\right\}$$

$$+(-)^{n-1}\left(\frac{2}{k}\right)^{2n-2}f^{2n+a-2}(o)\ \frac{\Gamma(2n+a+1)}{\Gamma(2n+a)}$$

$$\left.+(-)^{n-1}\left(\frac{2}{k}\right)^{2n}f^{2n+a}(o)\ \right]$$

$$b_{2n+1}=(-)^{n-1}\Gamma(2n+a+2)\left[\frac{b_1}{n!\,\Gamma(n+a+2)}-\frac{b_3}{(n-1)!\,\Gamma(n+a+3)}\right.$$

$$\left.+\ \cdots\ +(-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n+a-1}(o}{\Gamma(2n+a+1)}\ \right]$$

$$=(-)^{n+1}\left(\frac{2}{k}\right)^{1+a}\left[\Gamma(2n+a+2)f^{a+1}(o)\left\{\frac{1}{n!\,\Gamma(n+a+2)}\right.\right.$$

$$-\frac{\Gamma(a+4)}{(n-1)!\,\Gamma(n+a+3)\Gamma(a+3)}-\left(\frac{\Gamma(a+6)}{2!\,\Gamma(a+4)}\right)$$

$$\left.\left.-\frac{\Gamma(a+b)}{\Gamma(a+5)}\cdot\frac{\Gamma(a+4)}{\Gamma(a+3)}\right)\frac{1}{(n-2)!\,\Gamma(n+a+4)}-\ \cdots\ \right\}$$

$$|\left(\frac{2}{k}\right)^2\Gamma(2n+a+2)f^{a+2}(o)\left\{\frac{1}{(n-1)!\,\Gamma(n+a+3)}\right.$$

$$\left.-\frac{\Gamma(a+b)}{\Gamma(a+5)(n-2)!\,\Gamma(n+a+4)}-\ \cdots\ \right\}$$

$$+(-)^{n-2}\left(\frac{2}{k}\right)^{2n-4}f^{2n+a-3}(o)\left\{\frac{(2n+a+1)(2n+a)}{2!}\right.$$

$$\left.-(2n+a+1)(2n+a-1)\right\}+(-)^{n-1}\left(\frac{2}{k}\right)^{2n-2}f^{2n+a-2}(o).(2n+a+1)$$

$$\left.+(-1)^{n-1}f^{2n+a+1}(o)\left(\frac{2}{k}\right)^{2n}\right].$$

3. Next, let us take up the expansion in the form (II)

$$f(z)=\sum_{n=0}^{n=\infty} b_n \frac{J_{n+a}(kz)}{z^a}$$

where a is any positive integer or fraction.

Here we shall follow a method similar to that in § 2.

As, $$z^{-a}J_{n+a}(kz)=\left(\frac{k}{2}\right)^{n+a}\frac{z^n}{\Gamma(n+a+1)}\left[1-\left(\frac{k}{2}\right)^2\frac{z^2}{n+a+1}\right.$$
$$\left.+\left(\frac{k}{2}\right)^4\frac{z^4}{(n+a+1)(n+a+2)2!}-\ldots\right]$$

we get,

$$f(o)=\left(\frac{k}{2}\right)^a\frac{b_0}{\Gamma(a+1)}$$

$$f'(o)=\left(\frac{k}{2}\right)^{1+a}\frac{b_1}{\Gamma(a+2)}$$

$$\frac{f''(o)}{2!}=-\left(\frac{k}{2}\right)^{2+a}\frac{b_0}{\Gamma(a+2)}+\left(\frac{k}{2}\right)^{2+a}\frac{b_2}{\Gamma(a+3)}$$

$$\frac{f'''(o)}{3!}=-\left(\frac{k}{2}\right)^{3+a}\frac{b_1}{\Gamma(a+3)}+\left(\frac{k}{2}\right)^{3+a}\frac{b_3}{\Gamma(a+4)}$$

$$\frac{f^{2n}(o)}{2n!}=(-)^n\left(\frac{k}{2}\right)^{2n+a}\left[\frac{b_0}{n!\,\Gamma(n+a+1)}-\frac{b_2}{(n-1)!\,\Gamma(n+a+2)}\right.$$
$$\left.+\ldots+(-)^n\frac{b_{2n}}{\Gamma(2n+a+1)}\right.$$

$$\frac{f^{2n+1}(o)}{(2n+1)!}=(-)^n\left(\frac{k}{2}\right)^{2n+a+1}\left[\frac{b_1}{n!\,\Gamma(n+a+2)}\right.$$
$$\left.-\frac{b_3}{(n-1)!\,\Gamma(n+a+3)}+(-)^n\frac{b_{2n+1}}{\Gamma(2n+a+2)}\right].$$

From the above identities, we have

$$b_0=\left(\frac{2}{k}\right)^a\Gamma(a+1)\,f(o)$$

$$b_1=\left(\frac{2}{k}\right)^{1+a}\Gamma(a+2)\,.\,f'(o)$$

$$b_2 = \left(\frac{2}{k}\right)^{2+a}\Gamma(a+3)\left[\left(\frac{k}{2}\right)^{2+a}\frac{b_0}{\Gamma(a+2)} + \frac{f''(o)}{2!}\right]$$

$$= \left(\frac{2}{k}\right)^{a}\Gamma(a+3\left[\frac{\Gamma(a+1)}{\Gamma(a+2)}f(o) + \left(\frac{2}{k}\right)^{2}\frac{f''(o)}{2!}\right]$$

$$b_3 = \left(\frac{2}{k}\right)^{3+a}\Gamma(a+4\left[\left(\frac{k}{2}\right)^{3+a}\frac{b_1}{\Gamma(a+3)} + \frac{f''(o)}{3!}\right]$$

$$= \left(\frac{2}{k}\right)^{a+1}\Gamma(a+4)\left[\frac{\Gamma(a+2)}{\Gamma(a+3)}f'(o) + \left(\frac{2}{k}\right)^{2}\frac{f'''(o)}{3!}\right]$$

... ...

$$b_{2n} = (-)^{n+1}\Gamma(2n+a+1)\left[\frac{b_0}{n!\,\Gamma(n+a+1)} - \frac{b_2}{(n-1)!\,\Gamma(n+a+2)}\right.$$

$$\left. + \ldots + (-)^{n-1}\frac{f^{2n}(o)}{2n!}\right]$$

$$= (-)^{n+1}\left(\frac{2}{k}\right)^{a}\Gamma(2n+a+1)\left[f(o)\left\{\frac{\Gamma(a+1)}{n!\,\Gamma(n+a+1)}\right.\right.$$

$$\left. - \frac{\Gamma(a+3\,\Gamma\,a+1)}{(n-1)!\,\Gamma(a+2)\Gamma(n+a+2)} + \ldots\right\}$$

$$- \left(\frac{2}{k}\right)^{2}f''(o)\left\{\frac{\Gamma(a+3)}{(n-1)!\,2!\,\Gamma(n+a+2)} - \ldots\right\}$$

$$\left. + \ldots + (-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n}(o)}{2n!}\right]$$

$$b_{2n+1} = (-1)^{n+1}\left(\frac{2}{k}\right)^{1+a}\Gamma.\,2n+a+2)\left[f(o)\left\{\frac{\Gamma(a+2)}{n!\,\Gamma\,n+a+2)}\right.\right.$$

$$\left. - \frac{\Gamma(a+4)\Gamma(a+2)}{(n-1)!\,\Gamma(a+3)\Gamma(a+n+3)} + \ldots\right\}$$

$$- \left(\frac{2}{k}\right)^{2}f'''(o)\left\{\frac{\Gamma(a+4)}{3!\,(n-1)!\,\Gamma(n+a+3)} - \ldots\right\}$$

$$\left. + \ldots + (-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{f^{2n+1}(o)}{(2n+1)!}\right].$$

These co-efficients can be made to take simpler forms if we introduce the functions $O_1(\ )$, $O_2(\cdot)$, $O_3(\cdot)$, etc., defined by the following formulae :—

$$O_{2n}(z)=(-)^{n+1}\left(\frac{2}{k}\right)^{a}\Gamma(2n+a+1)\left[\frac{1}{2}\left\{\frac{\Gamma(a+1)}{n!\,\Gamma(n+a+1)}\right.\right.$$

$$\left.-\ \ldots\ \right\}-\left(\frac{2}{k}\right)^{2}\frac{1}{z^{2}}\left\{\frac{\Gamma(a+3)}{(n-1)!\,2!\,\Gamma(n+a+2)}-\ \ldots\ \right\}$$

$$+\ \ldots\ +(-)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{2n!}{2n!\,z^{2n+1}}\ \Bigg]$$

$$O_{2n+1}(z)=(-)^{n+1}\left(\frac{2}{k}\right)^{1+a}\Gamma(2n+a+2)\left[\frac{1}{z^{2}}\left\{\frac{\Gamma(a+2)}{n!\,\Gamma(n+a+2)}\right.\right.$$

$$\left.-\ \ldots\ \right\}-\frac{1}{z^{4}}\left(\frac{2}{k}\right)^{2}\left\{\frac{\Gamma(a+4)}{3!\,(n-1)!\,\Gamma(n+a+2)}-\ \ldots\ \right\}$$

$$+(-1)^{n-1}\left(\frac{2}{k}\right)^{2n}\frac{(2n+1)!}{(2n+1)!\,z^{2n+2}}\ \Bigg]$$

for then it is easily seen that b_{2n} is equal to the residue of the function $O_n(t)f(t)$ at the point $t=o$.

We have thus the general expansion

$$f(z)=\sum_{n=o}^{\infty}b_n J_{n+a}(kz)\ .\ z^{-a}$$

where $b_0=\left(\frac{2}{k}\right)^{a}\Gamma(a+1)f(o)$ and $b_n(n>o)$ is the residue of the function $O_n(t)f(t)$ at the point $t=o$, so that $b_n=\frac{1}{2\pi i}\int_{\alpha}O_n(t)\,f(t)dt$ where α is any simple contour surrounding the origin.

4. Some interesting examples can be given in illustration of the above expansions if we confine our attention to simple functions. A few of these are given below.

(*i*) Let us take as an example the expansion of $\cos z$. Let $k=1$.

Form I.

Here $f(o)=1,\ f'(o)=o,\ f''(o)=-1,\ f'''(o)=o$, etc.

$$b_0=1.\ b_2=2!\left[-\frac{2^2}{2!}+\frac{1}{1!}\right]=-2,\ b_4=4!\left[\frac{2}{4!}-\frac{1}{2!\,2!}-\frac{2}{3!}\right]$$

$=2$, etc.

Hence $\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - \ldots$

Form II.

Case (1). Let $a = \frac{1}{2}$

Then $b_0 = 2^{\frac{1}{2}}\Gamma(\frac{3}{2}) = \frac{\sqrt{2\pi}}{2}$, $b_1 = 0$,

$$b_2 = \sqrt{2}\,\Gamma(3+\tfrac{1}{2})\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} - 2^2\cdot\tfrac{1}{2}\right] = -\tfrac{5}{2}\sqrt{2\pi},$$

$$b_3 = 0,$$

$$b_4 = -\sqrt{2}\,\Gamma(5+\tfrac{1}{2})\left[\left\{\frac{\Gamma(\frac{3}{2})}{2!\,\Gamma(3+\frac{1}{2})} - \frac{\Gamma(3+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2+\frac{1}{2})\Gamma(4+\frac{1}{2})}\right\} + 2^2\frac{\Gamma(3+\frac{1}{2})}{\Gamma(4+\frac{1}{2})2!} - 2^4\cdot\frac{1}{4!}\right]$$

$$= \tfrac{9}{2}\sqrt{2\pi}.$$

Hence $z^{\frac{1}{2}}\cos z = \sqrt{2\pi}\left[\frac{1}{2}J_{\frac{1}{2}}(z) - \frac{5}{2}J_{\frac{5}{2}}(z) + \frac{9}{2}J_{\frac{9}{2}}(z) - \ldots\right]$.

Case (2), Let $a = \frac{1}{3}$.

Here $b_0 = 2^{\frac{1}{3}}\Gamma(\frac{1}{3}) = 2^{\frac{1}{3}}\cdot\frac{1}{3}\Gamma(\frac{1}{3})$

$$b_2 = 2^{\frac{1}{3}}\,\Gamma(3+\tfrac{1}{3})\left[\frac{\Gamma(1+\frac{1}{3})}{\Gamma(2+\frac{1}{3})} - 2^2\cdot\frac{1}{2!}\right] = -2^{\frac{1}{3}}\,\tfrac{5}{27}\Gamma(\tfrac{1}{3})$$

$$b_4 = -2^{\frac{1}{3}}\,\Gamma(5+\tfrac{1}{3})\left[\left\{\frac{\Gamma(1+\frac{1}{3})}{2!\,\Gamma(3+\frac{1}{3})} - \frac{\Gamma(3+\frac{1}{3})\Gamma(1+\frac{1}{3})}{\Gamma(2+\frac{1}{3})\Gamma(4+\frac{1}{3})}\right\} + 2^2\cdot\frac{\Gamma(3+\frac{1}{3})}{\Gamma(4+\frac{1}{3})2!} - 2^4\cdot\frac{1}{4!}\right] = 2^{\frac{1}{3}}\,\frac{1430}{729}\,\Gamma(\tfrac{1}{3}).$$

Hence $z^{\frac{1}{3}}\cos z = 2^{\frac{1}{3}}\,\Gamma(\frac{1}{3})\left[\frac{1}{3}J_{\frac{1}{3}}(z) - \frac{35}{27}J_{\frac{7}{3}}(z) + \frac{1430}{729}J_{\frac{13}{3}}(z) - \ldots\right]$

(*ii*) Now let us take $f(z) = z^n$. Let $k=1$

Here $f(o)=o$, $f'(o)=o$, $f''(o)=o$ ... $f^{n-1}(o)=o$,

$f^n(o) = n!$, $f^{n+r}(o) = o$ $(r>o)$.

$b_0 = 2^n \cdot n!$

$b_1 = 0$

$b_2 = 2^n (n+2) \cdot n!$

$b_3 = 0$

$$b_4 = -2^n \cdot n! \left\{ \frac{(n+3)(n+4)}{2!} - (n+4).(n+2) \right\}$$

$$= 2^n \cdot n! \frac{(n+4)(n+1)}{2!}.$$

$$b_6 = 2^n \cdot n! \left\{ \frac{(n+6)!}{3!(n+3)!} - \frac{(n+2)(n+6)!}{2!(n+4)!} + \frac{(n+4)(n+1)(n+6)!}{2!(n+5)!} \right\}$$

$$= 2^n \cdot n! \frac{(n+6)(n+2)(n+1)}{3!}$$

Hence $= 2^n \cdot n! \left\{ J_n(z) + (n+2) J_{n+2}(z) + \frac{(n+4)(n+1)}{2!} J_{n+4}(z) + \frac{(n+6)(n+2)(n+1)}{3!} J_{n+6}(z) + \ldots \right\}$

(*iii*) Let us now consider the function e^z.

Form I. Let $k=1$

Here $b_0 = 2^0 \cdot f(0) = 1$

$b_1 = 2 \cdot f(0) = 2$

$b_2 = [2 \cdot 1 + 2^2 \cdot 1] = 6$

$b_3 = 2 \cdot [3 \cdot 1 + 2 \cdot 1] = 14$

$$b_4 = -\left[\frac{3 \cdot 4}{2} - 4 \cdot 2 - 4 \cdot 2^2 - 2^4 \right] = 34.$$

Hence $e^z = J_0(z) + 2J_1(z) + 6J_2(z) + 14J_3(z) + 34J_4(z) + \ldots$

Form II. Let $a=\frac{1}{2}$, $k=1$

$$b_0=2^{\frac{1}{2}}\ \Gamma(\tfrac{1}{2})=\frac{\sqrt{2\pi}}{2}.$$

$$b_1=2^{\frac{3}{2}}\ \Gamma(\tfrac{3}{2})=\sqrt{2\pi}\cdot\frac{1}{2}.$$

$$b_2=2^{\frac{5}{2}}\ \Gamma(\tfrac{5}{2})\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}+2^2\cdot\frac{1}{2!}\right]=\sqrt{2\pi}.5$$

$$b_3=2^{\frac{7}{2}}\ \Gamma(\tfrac{7}{2})\left[\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{7}{2})}+2^2\cdot\frac{1}{3!}\right]=14\sqrt{2\pi}.$$

Hence $x^{\frac{1}{2}}e^x=\sqrt{2\pi}\,.\,J_{\frac{1}{2}}(x)+\frac{3}{2}J_{\frac{3}{2}}.(x)+5J_{\frac{5}{2}}(x)+14J_{\frac{7}{2}}(x)+\ldots]$.

(*iv*) Let us now expand $\sin z$. Let $k=1$.

Here $b_0=0$.

$b_1=2$.

$b_2=0$.

$b_3=8\,.\,3\ \ 2\,.\,1-\frac{1}{6}]=-2$,

$b_4=0$,

$$b_5=5!\left[\frac{32}{5!}-\frac{2}{2!\,3!}-\frac{2}{3!\,4}\right]=2.$$

Hence $\sin z=2J_1(z)-2J_3(z)+2J_5(z)-\ldots$.

5. The same method may be applied to expand any function $f(z)$ in the form

$$f(z)=\sum_{n=0}^{\infty}C_nJ_{na}(kz)$$

where a is an integer.

Now $J_{na}(kz)=\left(\frac{k}{2}\right)^{na}\frac{z^{na}}{\Gamma(na+1)}\left[1-\left(\frac{k}{2}\right)^2\frac{z^2}{na+1}\right.$

$$\left.+\left(\frac{k}{2}\right)^4\frac{z^4}{(na+1)(na+2)}-\ldots\right].$$

Hence $f(o)=C_0$,

$$f'(o)=o,\ f'''(o)=o,\ f^{v}(o)=o,\ \ldots$$

$$\frac{f''(o)}{2!}=-\left(\frac{k}{2}\right)^2\frac{C_0}{\Gamma(2)},\quad \frac{f^{iv}(o)}{4!}=\left(\frac{k}{2}\right)^4\frac{C_0}{\Gamma(3)},$$

etc.

$$\frac{f^{2r}(o)}{2r!}=\left(\frac{k}{2}\right)^{2r}\frac{(-)^r C_0}{\Gamma(r+1)} \text{ where } 2r<a$$

a be even.

$$\frac{f^{a}(o)}{a!}=\left(\frac{k}{2}\right)^{a}\frac{C_1}{\Gamma(a+1)}+(-1)^{\frac{a}{2}}\frac{C_0}{\Gamma(\frac{a}{2}+1)}\left(\frac{k}{2}\right)^{a},$$

$$f^{a+1}(o)=o.$$

$$\frac{f^{a+2}(o)}{(a+2)!}=-\left(\frac{k}{2}\right)^{a+2}\frac{C_1}{\Gamma(a+2)}+(-1)^{\frac{a}{2}+1}\frac{C_0}{\Gamma(\frac{a}{2}+2)}\left(\frac{k}{2}\right)^{a+2},$$

etc.

$$\frac{f^{2a}(o)}{2a!}=\left(\frac{k}{2}\right)^{2a}\frac{C_2}{\Gamma(2a+1)}+(-1)^{a}\left(\frac{k}{2}\right)^{2a}\frac{C_1}{\Gamma(a+\frac{a}{2}+1)}$$

$$+(-)^{a}\left(\frac{k}{2}\right)^{2a}\frac{C_0}{\Gamma(a+1)},$$

etc.

$$\frac{f^{na}(o)}{na!}=\left(\frac{k}{2}\right)^{na}\frac{C_n}{\Gamma(na+1)}+(-)^{\frac{a}{2}}\left(\frac{k}{2}\right)^{na}\frac{C_{n-1}}{\Gamma\{(n-1)a+\frac{a}{2}+1)\}}$$

$$+\ \ldots\ +(-)^{\frac{na}{2}}\left(\frac{k}{2}\right)^{na}\frac{C_0}{\Gamma(\frac{na}{2}+1)}.$$

Hence $C_0=f(o)$,

$$C_n=\left(\frac{2}{k}\right)^{na}\frac{f^{na}(o)}{na!}+(-)^{\frac{a}{2}+1}\frac{\Gamma(na+1)}{\Gamma\{(n-1)a+\frac{a}{2}+1)\}}C_{n-1}+\ldots$$

$$+(-)^{\frac{na}{2}}\frac{C_0\Gamma(na+1)}{\Gamma(\frac{na}{2}+1)}$$

If a be odd,

$$f(o)=C_0,$$

$$\frac{f^a(o)}{a!}=\left(\frac{k}{2}\right)^a\frac{C_1}{\Gamma(a+1)},$$

$$\frac{f^{2a}(o)}{2a!}=\left(\frac{k}{2}\right)^{2a}\frac{C_2}{\Gamma(2a+1)}+(-)^a\frac{C_0}{\Gamma(a+1)}\left(\frac{k}{2}\right)^{2a},$$

$$\frac{f^{3a}(o)}{3a!}=\left(\frac{k}{2}\right)^{3a}\frac{C_3}{\Gamma(3a+1)}+(-)^{3a}\left(\frac{k}{2}\right)^{3a}\frac{C_1}{\Gamma(2a+1)},$$

etc.

Hence $C_0=f(o)$,

$$C_1=\left(\frac{2}{k}\right)^a f^a(o)$$

$$C_2=\left(\frac{2}{k}\right)^{2a}\Gamma(2a+1)\left\{\frac{f^{2a}(o)}{2a!}+(-)^{a+1}\left(\frac{k}{2}\right)^{2a}\frac{f(o)}{\Gamma(a+1)}\right\},$$

$$C_3=\left(\frac{2}{k}\right)^{3a}\Gamma(3a+1)\left\{\frac{f^{3a}(o)}{3a!}+(-)^{a+1}\left(\frac{k}{2}\right)^{2a}\frac{f^a}{\Gamma(2a+1)}\right\},$$

etc.

Some more examples are given below :—

(1) $x^{\frac{1}{3}}e^x=2^{\frac{1}{3}}\Gamma(\frac{1}{3})[\frac{1}{3}J_{\frac{1}{3}}(x)+\frac{8}{9}J_{\frac{4}{3}}(x)+\frac{7\cdot7}{2!3^2}J_{\frac{7}{3}}(x)+\frac{1\cdot8\cdot4\cdot0}{2!3^3}J_{\frac{10}{3}}(x)+\ldots]$

(2) $x^{\frac{1}{4}}e^x=2^{\frac{1}{4}}\Gamma(\frac{1}{4})[\frac{1}{4}J_{\frac{1}{4}}(x)+\frac{5}{8}J_{\frac{5}{4}}(x)+\frac{6\cdot5}{3!2^3}J_{\frac{9}{4}}(x)+\frac{3\cdot3\cdot5}{6!4^3}J_{\frac{13}{4}}(x)+\ldots]$

(3) $z^{\frac{1}{4}}\cos z=2^{\frac{1}{4}}\Gamma(\frac{1}{4})[\frac{1}{4}J_{-\frac{1}{4}}(z)-\frac{3\cdot7}{2!2^3}J_{\frac{9}{4}}(z)+\frac{5\cdot9\cdot5}{5!2^5}J_{\frac{17}{4}}(z)-\ldots]$

(4) $z^{\frac{1}{5}}\cos z=2^{\frac{1}{5}}\Gamma(\frac{1}{5})[\frac{1}{5}J_{\frac{1}{5}}(z)-\frac{7\cdot7}{1!2^5}J_{\frac{11}{5}}(z)+\frac{2\cdot4\cdot0\cdot0}{3!2^5}J_{\frac{21}{5}}(z)-\ldots]$.

A proof of the validity of the expansions given in this paper may be obtained by adopting the method given in Whittaker's *Modern Analysis*, Art. 17.81, pp. 368-369. The detail discussion of this question is reserved for a subsequent paper.

Before I conclude I should like to express my indebtedness to Prof. S. K. Banerjee at whose suggestion I took up and under whose guidance I carried on the above investigation.

A note on the theory of the magnetometer[1]

By

L. Srivastava

The present paper contains the results of my investigations on magnetometric formulae (Gauss' A and B positions). In the usual Text Books the field round the *deflected needle* is taken uniform, *viz.*, the same at the centre as at the ends and the formulae are deduced. Here in this paper I have started with the inverse square law between the poles and thus got a system of forces which including Hm and Hm due to Earth's field are six in number. The formulae (5) and (5a) are the same as the usual Text Book formulae and it is found that, in order that these may form a very close approximation to actual relations, the deflections in A position should be 26·5° and in B position 31°.

In actual practice, however, it is found advantageous to use the sine position and not the Gauss' A and B positions and to use the empirical formulae

$$\frac{2mm'}{r^3}(1+Pr^{-2}+Qr^{-4})$$

for the couple and

$$\frac{m}{H}=\frac{r^3}{2}\sin\theta(1+Pr^{-2}+Qr^{-4})$$

for the raito $\frac{m}{H}$. P and Q are the distribution constants found by deflections for various values of r. Reference may be made to Lamont, *Proc. Roy. Soc.*, Vol. LXV, 1899.

Consider a small magnet A B free to rotate about its centre O in a horizontal plane, and let CD be the deflecting magnet (centre O').

Let AB=2a, CD=2c, O'O=x and $\angle AON=\theta$, N and S denote the north and south directions.

Also let $\pm m$ be the pole strengths of A, B and
" " $\pm m'$ " " " C, D and let H be the horizontal intensity of the Earth's field.

[1] Communicated by Prof. S. K. Banerjee.

There are six forces P, Q, R, S, Hm, Hm in the system (acting in the directions CA, AD, BC, DB, AN, BS. Now if p, q, r, s denote the perpendiculars from O on P, Q, R, S repectively, we get the condition of equilibrium by taking moments about O. Thus in Gauss' (A) position,

$$Pp-Qq+Rr-Ss-Hm.2a\sin\theta=0. \quad \ldots (1)$$

But $P.p=\frac{mm'}{AC^2}.p=\frac{mm'}{AC^3}.p.AC=\frac{mm'}{AC^3}2\Delta ADC$

$$=\frac{mm'a.\overline{x-c}\sin\overline{90+\theta}}{\{a^2+\overline{x-c}^2-a.\overline{x-c}\cos\overline{90+\theta}\}^{\frac{3}{2}}}$$

$$=\frac{mm'.a.\overline{x-c}\cos\theta}{(a^2+\overline{x-c}^2+2a.\overline{x-c}\sin\theta)^{\frac{3}{2}}}$$

$$=mm'\frac{d}{d\theta}\left(\frac{1}{(a^2+\overline{x-c}^2+2a.\overline{x-c}\sin\theta)^{\frac{1}{2}}}\right)$$

Similarly, by proper change in the signs of a and c, we get

$$Qq=-mm'\frac{d}{d\theta}\left(\frac{1}{(a^2+x+c^2+2a.x+c.\sin\theta)^{\frac{1}{2}}}\right).$$

Also $$Rr=mm'\frac{d}{d\theta}\left(\frac{1}{(a^2+\overline{x-c}^2-2a.\overline{x-c}\sin\theta)^{\frac{1}{2}}}\right)$$

and

$$Ss=mm'\frac{d}{d\theta}\left\{\frac{1}{a^2+\overline{x+c}^2-2a.\overline{x+c}\sin\theta)^{\frac{1}{2}}}\right\}.$$

Substituting these values of Pp, Qq, Rr, Ss in (1) we get,

$$mm'\frac{d}{d\theta}\left[\frac{-1}{(a^2+\overline{x-c}^2+2a.\overline{x-c}\sin\theta)^{\frac{1}{2}}}+\frac{1}{(a^2+x+c^2+2a.x+c\sin\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\frac{1}{(a^2+x-c^2-2a.x-c\sin\theta)^{\frac{1}{2}}}-\frac{1}{(a^2+x+c^2-2a.x+c\sin\theta)^{\frac{1}{2}}}\right]$$

$$=Hm.2a\sin\theta \quad \ldots (2)$$

Now putting $\frac{a}{x-c}=h$, $\frac{a}{x+c}=h'$ and $\sin\theta=\mu$, the left hand side expression in (2) becomes

$$mm'\frac{d}{d\theta}\Big[-\frac{-1}{x-c}\frac{}{(2+2\mu h+h^2)^{\frac{1}{2}}}+\frac{1}{x+c}\frac{}{(1+2\mu h'+h'^2)^{\frac{1}{2}}}$$

$$+\frac{1}{x-c}\frac{}{(1-2\mu h+h^2)^{\frac{1}{2}}}-\frac{1}{x-c}\frac{}{(1-2\mu h'+h'^2)^{\frac{1}{2}}}\Big].$$

Expanding the radicals and putting

$$\frac{1}{(1-2\mu h+h^2)^{\frac{1}{2}}}=1+P_1h+P_2h^2+\ldots+P_nh^n+\ldots$$

where P_1, $P_2\ldots P_n$ are the Legendre's co-efficients (or Zonal Harmonics) and are functions of μ only. We get the left hand expression of (2)

$$=mm'\frac{d}{d\theta}\Big[-\frac{1}{x+c}(1-P_1h+P_2h^2-P_3h^3+\ldots)$$

$$+\frac{1}{x+c}(1-P_1h'+P_2h'^2+\ldots)+\frac{1}{x-c}(1+P_1h+P_2h^2+\ldots)$$

$$-\frac{1}{x+c}(1+P_1h'+P_2h'^2+\ldots)\Big]$$

$$=\frac{mm'd}{d\theta}\Big[\frac{2}{x-c}\Big(P_1h+P_3h^3+\ldots\Big)-\frac{2}{x+c}\Big(P_1h'+P_3h'^3+\ldots\Big)\Big]$$

$$=2mm'\frac{d}{d\theta}\Big[P_1\Big(\frac{h}{x-c}-\frac{h'}{x+c}\Big)+P_3\Big(\frac{h^3}{x-c}-\frac{h'^3}{x+c}\Big)+\ldots\Big].$$

So equation (2) becomes

$$2mm'\frac{d}{d\theta}\Big[P_1\Big(\frac{h}{x-c}-\frac{h'}{x+c}\Big)+P_3\Big(\frac{h^3}{x+c}-\frac{h'^3}{x+c}\Big)+\ldots\Big]$$

$$=\Pi m\cdot 2a\sin\theta.$$

Now putting for h and h' their values, and then dividing by $2am$, we have

$$m'\left[\left\{\frac{1}{(x-c)^2}-\frac{1}{(x+c)^2}\right\}\frac{dP_1}{d\theta}+\left\{\frac{a^2}{(x-c)^4}-\frac{a^2}{(x+c)^4}\right\}\frac{dP_3}{d\theta}\right.$$

$$\left.+\ \ldots\ \right]=H\sin\theta\ \ \ldots\ \ (3)$$

Now, if a be small compared with $(x-c)$ or $(x+c)$, the left hand series in 3) is rapidly convergent, and obviously the 1st term alone is a close approximation. Retaining the 1st term only, we get

$$m'\left[\left\{\frac{1}{(x-c)^2}-\frac{1}{(x+c)^2}\right\}\frac{dP_1}{d\theta}\right]=H\sin\theta\ \ldots\qquad\ldots\ \ (4)$$

but P_1 is easily seen to be equal to μ or $\sin\theta$ here. Therefore

$$\frac{dP_1}{d\theta}=\cos$$

Thus from (4)

$$m'\left[\left\{\frac{1}{(x-c)^2}-\frac{1}{(x+c)^2}\right\}\cos\theta\right]=H\sin\theta,$$

$$\frac{m'\cdot 4c}{(x^2-c^2)^2}\cos\theta=H\sin\theta,$$

Putting $2m'c$ (magnetic moment of the magnet CD)$=M$, we get

$$\frac{M}{H}=\frac{(x^2-c^2)^2}{2c}\tan\theta\ \ldots\qquad\ldots\ \ (5)$$

It is known that Legendre's Co-efficient are given by the formula

$$P_n=\frac{1}{2^n\cdot\lfloor n}\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n.$$

Therefore,

$$P_3=\frac{1}{2^3\lfloor 3}\left(\frac{d}{d\mu}\right)^3(\mu^2-1)$$

$$=\frac{5\mu^3-3\mu}{2}$$

Therefore, $\frac{dP_3}{d\theta}=\frac{dP_3}{d\mu}\cdot\frac{d\mu}{d\theta}=\frac{1}{2}\cos\theta\,[15\mu^2-3]$.

If then $\mu^2=\frac{1}{5}$, the second term of (3) including $\frac{dP_3}{d\theta}$ vanishes and the first term constitutes a still higher degree of approximation. But $\mu^2=\frac{1}{5}$ corresponds to $\sin^2\theta=\frac{1}{5}$ or $\sin\theta=\frac{1}{\sqrt{5}}=\sin 26^\circ.5$ nearly, which gives $\theta=26^\circ.5$.

This deflection then is a good one to obtain, when determing H by Gauss' (A) position.

Next we take Gauss' (B) position.

Here again we use the same terminology, and in addition, put, $OD=OC=(x^2+c^2)^{\frac{1}{2}}\equiv l$ say and $\frac{a}{l}=\sin\alpha$, and taking moments about O, we get

$$Pp-Qq+Rr-Ss+Hm.2a\sin\theta=0\ldots \qquad \ldots \quad (1a)$$

Now $Pp=\frac{mm'}{AC^2}\cdot p=\frac{mm'}{AC^3}\cdot p\cdot AC=\frac{mm'}{AC^3}\cdot 2\Delta AOC.$

$$=\frac{mm'al\sin(\pi-\overline{\theta-\alpha})}{[a^2+l^2-2al\cos(\pi-\overline{\theta-\alpha})]^{\frac{3}{2}}}=\frac{mm'al\sin(\theta-\alpha)}{\{a^2+l^2+2al\cos(\theta-\alpha)\}^{\frac{3}{2}}}$$

$$=mm'\frac{d}{d\theta}\left\{\frac{1}{(a^2+l^2+2al\cos\theta-\alpha)^{\frac{1}{2}}}\right\}$$

and by suitable alterations in the signs of a and α, we get

$$Qq=\frac{mm'al\sin(\theta+\alpha)}{[a^2+l^2+2al\cos(\theta+\alpha)]^{\frac{3}{2}}}=mm'\frac{d}{d\theta}\left\{\frac{1}{[a^2+l^2+2al\cos(\theta+\alpha)]^{\frac{1}{2}}}\right\}$$

$$Rr=\frac{mm'al\sin(\theta-\alpha)}{[a^2+l^2-2al\cos(\theta-\alpha)]^{\frac{3}{2}}}=mm'\frac{d}{d\theta}\left\{\frac{-1}{[a^2+l^2-2al\cos(\theta-\alpha)]^{\frac{1}{2}}}\right\}$$

and $Ss=\frac{mm'al\sin(\theta+\alpha)}{[a^2+l^2-2al\cos(\theta+\alpha)]^{\frac{3}{2}}}=mm'\frac{d}{d\theta}\left\{\frac{-1}{[a^2+l^2-2al\cos(\theta+\alpha)]^{\frac{1}{2}}}\right\}$.

Substituting the values of Pp, Qq, etc., in $(1\cdot a)$ we have

$$m'\frac{d}{d\theta}\Big[\frac{1}{(a^2+l^2+2al\cos\theta-a)^{\frac{1}{2}}}-\frac{1}{a^1+l^2+2al\cos\overline{\theta+a})^{\frac{1}{2}}}$$

$$-\frac{1}{\{a^2+l^2-2al\cos(\theta-a)\}^{\frac{1}{2}}}+\frac{1}{(a^2+l^2+2al\cos\overline{\theta+a})^{\frac{1}{2}}}\Big]$$

$$=-\mathrm{H}m\cdot 2a\sin\theta-\ldots\qquad\ldots\quad(2a)$$

Putting $\frac{a}{l}=h$, $\cos(\theta+a)=\mu$, and $\cos(\theta-a)=\mu'$, we get

$$mm'\frac{d}{d\theta}\Big[\frac{1}{l(1+2\mu'h+h^2)^{\frac{1}{2}}}-\frac{1}{l(1+2\mu h+h^2)^{\frac{1}{2}}}$$

$$-\frac{1}{l\,1-2\mu h'+h^2)^{\frac{1}{2}}}+\frac{1}{l(1-2\mu h+h^2)^{\frac{1}{2}}}\Big]$$

$$=-\mathrm{H}m\cdot 2a\sin\theta.$$

$$\frac{mm'}{l}\frac{d}{d\theta}[(1-\mathrm{P}'_1h+\mathrm{P}'_2h^2-\mathrm{P}'_3h^3+\ldots)-(1-\mathrm{P}_1h+\mathrm{P}_2h^2-\mathrm{P}_3h^3$$

$$+\ldots)-(1+\mathrm{P}'_1h+\mathrm{P}'_2h^2+\ldots)+(1+\mathrm{P}_1h+\mathrm{P}_2h^2+\ldots)]$$

$$=-\mathrm{H}m.2a\sin\theta,$$

or

$$\frac{mm'}{l}\frac{d}{d\theta}[2\,\mathrm{P}'_1h+\mathrm{P}'_3h^3+\ldots)-2(\mathrm{P}_1h+\mathrm{P}_3h^3+\ldots)]=\mathrm{H}m.2a\sin\theta$$

or

$$-\frac{2mm'}{l}\frac{d}{d\theta}[(\mathrm{P}'_1-\mathrm{P}_1)+(\mathrm{P}'_3-\mathrm{P}_3)h^2+\ldots]=\mathrm{H}m\cdot 2a\sin\theta.$$

Now putting for h its value $\frac{a}{l}$, and dividing both sides by $-2am$, we get

$$\frac{m'}{l^2}\frac{d}{d\theta}\Big[(\mathrm{P}'_1-\mathrm{P}_1)+(\mathrm{P}'_3-\mathrm{P}_3)\frac{a^2}{l^2}+\ldots\Big]=\mathrm{H}\sin\theta\ldots.\qquad\ldots\quad(3a)$$

Here again, if a is small compared with l, the series on the left hand side is rapidly convergent, and the first term alone constitutes a close approximation, thus

$$\frac{m'}{l^2}\frac{d}{d\theta}(P'_1-P_1)=H\sin\theta \ldots \qquad \ldots \quad (4\cdot a)$$

where obviously $P'_1=\mu'=\cos(\theta-a)$ and $P_1=\mu=\cos(\theta+a)$.

Hence $(4\cdot a)$ gives

$$\frac{m'}{l^2}(-\sin\overline{\theta-a}+\sin\overline{\theta+a})=H\sin\theta,$$

or

$$\frac{m'}{\ldots}\cdot 2\cos\theta\sin a=H\sin\theta,$$

$$\frac{m'}{l^2}\cdot 2\cos\theta\cdot\frac{c}{l}=H\sin\theta,$$

and if $M=2cm'$ (the magnetic moment of the magnet CD), we get

$$\frac{M}{H}=l^3\tan\theta=(c^2+c^2)^{\frac{3}{2}}\tan\theta \ldots \qquad \ldots \quad (5\cdot a)$$

Here, also, if the second term of $(3\cdot a)$ is made to vanish, the 1st term will constitute a still higher degree of approximation. The 2nd term of $(3\cdot a)$ will vanish if

$$\frac{d}{d\theta}(P'_3-P_3)=0 \ldots \qquad \ldots \quad (6\cdot a)$$

where $P'_3=\frac{5\mu'^3-3\mu'}{2}$, and $P_3=\frac{5\mu^3-3\mu}{2}$ and $\mu'=\cos(\theta-a)$, $\mu=\cos(\theta+a)$.

Substituting the values of P'_3 and P_3 in (6a) and differentiating we get,

$$(5\mu'^2-1)\frac{d\mu'}{d\theta}-(5\mu^2-1)\frac{d\mu}{d\theta}=0,$$

or

$$-[5\cos^2(\theta-a)-1]\sin(\theta-a)-[5\cos^2(\theta+a)-1]\sin(\theta+a)=0,$$

and on simplifying we get

$$[\sin(\theta-\alpha)-\sin(\theta+\alpha)][5\{\sin^2(\theta-\alpha)+\sin(\theta-\alpha)\sin(\theta+\alpha)+\sin^2(\theta+\alpha)\}-4]=0.$$

which gives either $\sin(\theta-\alpha)-\sin(\theta+\alpha)=0$,

[*i.e.*, $\alpha=0$ or $\theta=90$ which is not practical]

or $5\{\sin^2(\theta-\alpha)+\sin(\theta-\alpha)\sin(\theta+\alpha)+\sin^2(\theta+\alpha)\}-4=0$,

that is, $\sin^2\theta[3-4\sin^2\alpha]=[\frac{4}{5}-\sin^2\alpha]$.

or $\sin^2\theta=\dfrac{\frac{4}{5}-\sin^2\alpha}{3-4\sin^2\alpha}=\dfrac{4}{15}$ nearly when α is small.

Therefore, we get $\sin\theta=\dfrac{2}{\sqrt{15}}=\sin 31^\circ$ nearly.

Thus $\theta=31^\circ$, and this deflection then is a good one to obtain when finding H by Gauss' B position.

On the motion of an Elongated Spheroid in Viscous Fluid Media

By

Bholanath Pal.

It is well known that the problem of the motion of a sphere in a viscous fluid medium has been solved by various investigators. But the corresponding problem for a spheroid or an ellipsoid has been attempted with success by a few. In a previous paper[1], I investigated the motion of a spheroid of small ellipticity, *i.e.*, a spheroid slightly differing from a sphere, in a viscous fluid. In the present paper, I have used the same method to investigate the other extreme case, viz., the motion of a very elongated spheroid in a viscous liquid along its axis.

It is easy to see that an equation[2] of the type

$$r=a\{1+2(1-\epsilon)\,P_2\,(\cos\theta)\}$$

will represent approximately an elongated spheroid or very roughly a finite rod of length $(3-2\epsilon)\,a$, provided ϵ is an extremely small quantity, whose square and higher powers are negligible.

Let u, v, w be the three velocity-components and suppose $u=u+U$, where U is the velocity of the spheroid.

The differential equations for the motion of the fluid are

$$U\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,$$

$$U\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v.$$

$$U\frac{\partial w}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w.$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0;$$

[1] *Bulletin of the Cal. Math. Soc.*, Vol. X, No. 2, Sept., (1919).

[2] This method of representing an elongated spheroid was suggested to me by Prof. Banerji. He is using this equation to solve a number of interesting problems involving an elongated spheroid or a thin finite rod, for example, the problem of the electrical oscillations on an elongated spheroid or a thin rod, the radiation from the two ends of a finite rod, etc.

where p is the pressure, ρ the density, and ν the kinematic co-efficient of viscosity.

The corresponding auxiliary conditions are $u=0$, $v=0$, $w=0$, at infinity, and $u=-U$, $v=0$, $w=0$, on the surface.

From the equations of motion, we easily obtain

$$\nabla^2 p=0.$$

Assuming $$p=\rho U \frac{\partial \phi}{\partial x}, \text{ where } \nabla^2\phi=0,$$

we take

$$u=-\frac{\partial \phi}{\partial x}+u',$$

$$v=-\frac{\partial \phi}{\partial y}+v',$$

$$w=-\frac{\partial \phi}{\partial z}+w'.$$

Substituting this value of u in the first equation of the equations of motion we get

$$\left(\nabla^2-\frac{U}{\nu}\frac{\partial}{\partial x}\right)u'=0.$$

Writing k for $\frac{U}{2\nu}$, this can be written in the from

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)u'=0.$$

Similarly $$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)v'=0,$$

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)w'=0,$$

and $$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0.$$

The vortex lines will be circles having the axis of x as a common axis ; we may, therefore, assume

$$\xi=0, \quad \eta=-\frac{\partial X}{\partial z}, \quad \zeta=\frac{\partial X}{\partial y},$$

where X is a function of x and ω (the distance from the axis of x) only and ξ, η, ζ are components of vorticity.

Then X must satisfy the equatton

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)X=0.$$

Thus

$$2k\frac{\partial u'}{\partial x}=\frac{\partial^2 X}{\partial x^2}-2k\frac{\partial X}{\partial x},$$

$$2k\frac{\partial v'}{\partial x}=\frac{\partial^2 X}{\partial x\partial y},$$

$$2k\frac{\partial w'}{\partial x}=\frac{\partial^2 X}{\partial x\partial z}.$$

Therefore

$$u'=\frac{1}{2k}\frac{\partial X}{\partial x}-X,$$

$$v'=\frac{1}{2k}\frac{\partial X}{\partial y},$$

$$w'=\frac{1}{2k}\frac{\partial X}{\partial z}.$$

In the differential equation

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)X=0,$$

writing $e^{kx}X'$ for X, we obtain

$$(\nabla^2-k^2)\,X'=0$$

The solution[1] of this differential equation is

$$X'=\sum_0^\infty C'_n f_n(kr)\,r^n P_n$$

where

$$f_n(\zeta)=\left(\frac{d}{\zeta d\zeta}\right)^n\frac{e^{-\zeta}}{\zeta}$$

i.e.

$$f_n(\zeta)=(-1)^n\frac{e^{-\zeta}}{\zeta^{n+1}}\left[1+\frac{n(n+1)}{2\cdot\zeta}+\frac{(n-1)\,n\,(n+1)\,(n+2)}{2.4.\zeta^2}+\ldots\right.$$
$$\left.\ldots+\frac{1.2.3\ldots2n}{2.4.6\ldots2n\cdot\zeta^n}\right],$$

[1] See Prof. Lamb's Hydrodynamics, Third edition, p. 480.

Thus

$$f_0(\zeta)=\frac{e^{-\zeta}}{\zeta},\ f_1(\zeta)=-\left(\frac{1}{\zeta^2}2+\frac{1}{\zeta^3}\right)e^{-\zeta},\ f_2(\zeta)=\left(\frac{1}{\zeta^3}+\frac{3}{\zeta^4}+\frac{3}{\zeta^5}\right)e^{-\zeta},$$

etc.

Therefore

$$X'=\left[C_0+C_1\left\{1+\frac{\alpha}{r}\right\}P_1+C_2\left\{1+\frac{\beta}{r}+\frac{\gamma}{r^2}\right\}2P_2+\ldots\right]\frac{e^{-kr}}{r}$$

where α, β, γ...are all known constants such that

$$\alpha=\frac{1}{k}\ ,\ \beta=\frac{3}{k}\ ,\ \gamma=\frac{3}{k^2}\ \ldots$$

From the foregoing relation we get

$$X=\left[C_0+C_1\left\{1+\frac{\alpha}{r}\right\}P_1+C_2\left\{1+\frac{\beta}{r}+\frac{\gamma}{r^2}\right\}2P_2+\ldots\right]\frac{e^{-k(r-x)}}{r}$$

Taking kr very small and neglecting square of the velocity this can be written in the from

$$X=\left[C_0+C_1\left\{1+\frac{\alpha}{r}\right\}P_1+C_2\left\{1+\frac{\beta}{r}+\frac{\gamma}{r^2}\right\}2P_2+\ldots\right]\left\{\frac{1}{r}-k+\frac{kx}{r}+\ldots\right\}.$$

From the auxiliary conditions we can easily find

$$\phi=\frac{A_0}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ldots$$

where A_0, A_1, A_2...are different constants.

We have

$$u=-\frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial X}{\partial x}-X,$$

$$v=-\frac{\partial\phi}{\partial y}+\frac{1}{2k}\frac{\partial X}{\partial y},$$

$$w=-\frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial X}{\partial z}.$$

Thus finally we obtain

$$u=-\frac{\partial}{\partial x}\left\{\frac{A_0}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\dots\right\}$$

$$-\frac{C_0}{2k}\left\{\frac{4}{3}\frac{k}{r}-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\dots\right\}$$

$$-\frac{C_1}{2k}\left\{\frac{2k}{3r}-\frac{1}{3r^2}-\frac{6kr}{5}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right\}$$

$$+\frac{2}{3}r\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)-\frac{2}{3\cdot5}kr^3\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\Big\}$$

$$-\frac{aC_1}{2k}\left\{\frac{k}{3r^2}-\frac{9}{5}k\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{2}{3}kr\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right.$$

$$\left.-\frac{1}{5}kr^2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\right\}$$

$$+\frac{C_2}{2k}\left\{\frac{8}{15}\frac{k}{r}-\frac{8}{5}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{12}{5}kr\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right.$$

$$-\frac{26}{21}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{3}{5}r^2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)-\frac{2}{5}kr^3\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)$$

$$\left.-\frac{3}{5\cdot7}kr^4\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right\}$$

$$+\frac{\beta C_2}{2k}\left\{\frac{4}{15}\frac{k}{r^2}-\frac{4}{5}\frac{1}{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{8}{5}k\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right.$$

$$-\frac{37}{21}kr\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{4}{5}r\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)-\frac{3}{5}kr^2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)$$

$$\left.-\frac{4}{5\cdot7}kr^3\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right\}$$

$$+\frac{\gamma C_2}{2k}\left\{\frac{4}{5}\frac{k}{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{16}{7}k\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\right.$$

$$\left.-\frac{4}{5}kr\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)-\frac{1}{7}kr^2\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right\}+\dots$$

and similar two expressions for v and w.

The contants A_0, A_1, A_2...and C_0, C_1 C_2...are to be determined from the auxiliary conditions.

We have $u=-U$, when $r=a\{1+2(1-\epsilon)P_2\}$.

Therefore

$$-U=\frac{A_0}{3^3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}-\frac{A_1}{3^4a^3}\left\{\frac{9(1+3\epsilon)}{\mu^4}-\frac{3(1+6\epsilon)}{\mu^6}+\frac{3\epsilon}{\mu^8}\right\}$$

$$+\frac{3A_2}{3^5a^4}\left\{\frac{15(1+4\epsilon)}{\mu^5}-\frac{(9+56\epsilon)}{\mu^7}+\frac{12\epsilon}{\mu^9}\right\}-\frac{2C_0}{3^3a}\left\{\frac{3(1+\epsilon)}{\mu^2}-\frac{\epsilon}{\mu^4}\right\}$$

$$-\frac{C_0}{3^3 2ka^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}-\frac{C_0}{2.3^3a}\left\{9(1+\epsilon)-\frac{3(1+2\epsilon)}{\mu^2}+\frac{\epsilon}{\mu^4}\right\}+$$

$$-\frac{C_1}{3^3a}\left\{\frac{3(1+\epsilon)}{\mu^2}-\frac{\epsilon}{\mu^4}\right\}+\frac{C_1}{2k\cdot 3^4a^2}\left\{\frac{3(1+2\epsilon)}{\mu^4}-\frac{2\epsilon}{\mu^6}\right\}$$

$$-\frac{C_1}{5.3a}\left\{\frac{3(1+\epsilon)}{\mu}-\frac{\epsilon}{\mu^3}\right\}$$

$$+\frac{C_1}{2.3^3a}\left\{9(1+\epsilon)-\frac{3(1+2\epsilon)}{\mu^2}+\frac{\epsilon}{\mu^4}\right\}-\frac{C_1}{k.3^4a^2}\left\{\frac{9(1+2\epsilon)}{\mu^2}\right.$$

$$\left.-\frac{3(1+4\epsilon)}{\mu^4}+\frac{2\epsilon}{\mu^6}\right\}$$

$$-\frac{C_1}{5.3^3a}\left\{15(1+\epsilon)\mu-\frac{(9+14\epsilon)}{\mu}+\frac{3\epsilon}{\mu^3}\right\}+\ldots$$

$$-\frac{aC_1}{2.3^4a^2}\left\{\frac{3(1+2\epsilon)}{\mu^4}-\frac{2\epsilon}{\mu^6}\right\}-\frac{aC_1}{2.5.3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}$$

$$+\frac{aC_1}{3^4a^2}\left\{\frac{9(1+2\epsilon)}{\mu^2}-\frac{3(1+4\epsilon)}{\mu^4}+\frac{2\epsilon}{\mu^6}\right\}-\frac{aC_1}{2k.3^4a^3}\left\{\frac{9(1+3\epsilon)}{\mu^4}\right.$$

$$\left.-\frac{3(1+6\epsilon)}{\mu^6}+\frac{3\epsilon}{\mu^8}\right\}$$

$$-\frac{aC_1}{2.5.3^3a^2}\left\{\frac{15(1+2\epsilon)}{\mu}-\frac{(9+28\epsilon)}{\mu^3}+\frac{6\epsilon}{\mu^5}\right\}-\ldots$$

$$+\frac{4C_2}{5.3^3a}\left\{\frac{3(1+\epsilon)}{\mu^2}-\frac{\epsilon}{\mu^4}\right\}+\frac{4C_2}{5k.3^3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}$$

$$-\frac{2C_2}{5.3a}\left\{\frac{3(1+\epsilon)}{\mu}-\frac{\epsilon}{\mu^3}\right\}$$

$$-\frac{13C_2}{7.3^3a}\left\{9(1+\epsilon)-\frac{3(1+2\epsilon)}{\mu^2}+\frac{\epsilon}{\mu^4}\right\}-\frac{C_2}{2k.5.3a^2}\left\{\frac{15(1+2\epsilon)}{\mu}\right.$$

$$\left.-\frac{(9+28\epsilon)}{\mu^3}+\frac{6\epsilon}{\mu^5}\right\}$$

$$+\frac{C_2}{5.3a}\left\{15(1+\epsilon)\mu-\frac{(9+14\epsilon)}{\mu}+\frac{3\epsilon}{\mu^3}\right\}$$

$$-\frac{C_2}{5.7a}\left\{105(1+\epsilon)\mu^2-(90+125\epsilon)+\frac{(9+39\epsilon)}{\mu^2}-\frac{3\epsilon}{\mu^4}\right\}+\ldots$$

$$+\frac{2\beta C_2}{5.3^4a^2}\left\{\frac{3(1+2\epsilon)}{\mu^4}-\frac{2\epsilon}{\mu^6}\right\}+\frac{2\beta C_2}{5k.3^4a^3}\left\{\frac{3(1+3\epsilon)}{\mu^5}-\frac{3\epsilon}{\mu^7}\right\}$$

$$-\frac{4\beta C_2}{5.3^3a^2}\left\{\frac{3(1+2\epsilon)}{\mu^3}-\frac{2\epsilon}{\mu^5}\right\}-\frac{37\beta C_2}{14.3^4a^2}\left\{\frac{9(1+2\epsilon)}{\mu^2}\right.$$

$$\left.-\frac{(3+12\epsilon)}{\mu^4}+\frac{2\epsilon}{\mu^6}\right\}$$

$$-\frac{2\beta C_2}{5k.3^3a^3}\left\{\frac{15(1+3\epsilon)}{\mu^3}-\frac{(9+42\epsilon)}{\mu^5}+\frac{9\epsilon}{\mu^7}\right\}$$

$$+\frac{\beta C_2}{10.3a^2}\left\{\frac{15(1+2\epsilon)}{\mu}-\frac{(9+28\epsilon)}{\mu^3}+\frac{6\epsilon}{\mu^5}\right\}$$

$$-\frac{2\beta k}{5.7.3^2a^2}\left\{104(1+2\epsilon)-\frac{(90+250\epsilon)}{\mu^2}+\frac{(9+78\epsilon)}{\mu^4}-\frac{6\epsilon}{\mu^6}\right\}+\ldots$$

when μ is written for $\cos\theta$.

Now this must hold for all values of $\cos\theta$. Therefore equating to zero the co-efficients of various powers of $\frac{1}{\mu}$, we get a series of equations from which the values of the constants $C_0, C_1, C_2, \ldots$ and $A_0, A_1, A_2, \ldots$ can be determined.

Thus we have

$$U=\frac{(1+\epsilon)}{6a}C_0-\frac{(1+\epsilon)}{6a}C_1-\frac{\{(41a-14\beta)+(62a-48\beta)\epsilon\}}{21a^2}C_2+\ldots$$

$$0=\frac{(1+\epsilon)}{3a}C_1-\frac{(1+\epsilon)}{a}C_2+\ldots$$

$$0=\frac{(3+2\epsilon)}{2.3^2a}C_0+\frac{(3+4\epsilon)}{2.3^2a}C_1-\frac{\{(24a+175\beta)-(386a-630\beta)\epsilon\}}{5.14.3^2a^3}C_2+\ldots$$

$$0=\frac{(1+2\epsilon)}{3^2a^2}A_0-\frac{(1+2\epsilon)}{2k.3^2a^2}C_0-\frac{\epsilon}{k.3^2a^2}C_1-\frac{\{7+(20-3ka)\epsilon\}}{k.3^2a^2}C_2+\ldots$$

$$0=\frac{(1+3\epsilon)}{3^3a^3}A_1-\frac{\epsilon}{2.3^3a}C_0-\frac{\epsilon}{2.3^3a}C_1-\frac{\{105-(140-8ka)\epsilon\}}{2k.35.3^3a^2}C_2+\ldots$$

$$0=\frac{2\epsilon}{3^3a^2}A_0-\frac{5(1+4\epsilon)}{3^3a^4}A_2-\frac{\epsilon}{k.3^3a^2}C_0-\frac{14\epsilon}{k.3^3a^2}C_2+\ldots$$

... ...

... ...

Keeping C_0, C_1, C_2 and A_0, A_1, A_2 only and neglecting the square of the velocity the following approximate values of the constants can be easily obtained.

$$C_0=\frac{97}{102}\left\{1+\frac{15}{34}\epsilon\right\}Ua,$$

$$C_1=-\frac{35}{34}\left\{1-\frac{57}{34}\epsilon\right\}Ua,$$

$$C_2=-\frac{35}{102}\left\{1-\frac{57}{34}\epsilon\right\}Ua,$$

... ...

... ...

and

$$A_0=-\frac{131}{34}\left\{1+\frac{117}{34}\epsilon\right\}va,$$

$$A_1=-\frac{4\epsilon}{52}Ua^3,$$

$$A_2=0.$$

... ...

... ...

Thus ϕ and X can be found and u, v, w can be determined in terms of these two functions.

I am grateful to Prof. S. K. Banerjee for his valuable suggestion and encourgement.

Cullis's Matrices and Determinoids

By

Rai Bahadur A. C. Bose, M.A.

University of Calcutta—Readership Lectures: Matrices and Determinoids, by C. E. Cullis, M.A., Ph.D., Vol. I, i–xii, pages 1–430 (1913); Price 21 Shillings; Vol. II, i–xxiv, pages 1–555 (1918); Price 42 Shillings, Cambridge; at the University Press.

Fuchs: To study *Modern Algebra* I'm most persuaded.
Meph: It was not my wish to lead thee astray.
But as concerns this Science, truly
'Tis difficult to avoid the empty form,
And should'st thou lack clear comprehension,
Scarcely the indices thou'll know apart.
'Tis safest far to trust but *one*
And built upon your master's formulas.
On the whole—cling closely to your *symbols*.
Then, for the weal of research you may gain
An entrance to the formula's sure domain.
Fuchs: The symbol, it must lead to some result?
Meph: Granted. But never worry about results,
For, mind you, just when the results are wanting
A symbol at the nick of time appears.
To symbolic treatment all things yield,
Provided we stay in the general field.
Should a solution prove elusive,
Write the equation in determinant form.
Write what you please, but *never* calculate.
Symbols are patient and long suffering,
A single stroke completes the whole affair.
Symbols for every purpose do suffice.

Lasswitz, Kurd.

The Tragedy of Faust, &c., &c.

As Prof. C. R. Lanman of Harvard, speaking of the labours of a great oriental scholar, said:—

"Scholarship moves like the tides of the sea. It is started by some great celestial attraction, some force moving in an ecliptic high above the level world of letters (and of Science) and with gathering it comes to its flood."

Thanks to the enlightened interest and leadership of the Hon'ble Sir Asutosh Mookerjee, Kt., M.A., D.L., the President of the Post-Graduate Councils of Teaching in Arts and Science and to the band of devoted workers his genius has collected, the Calcutta University has already moved to an assured place in the world of letters and Science.

Foremost among the mathematicians working in the field of Pure Mathematics is Professor Cuthbert Edmund Cullis, M.A. (Cantab.), Ph.D. (Jena), formerly Smith's Prizeman of the Cambridge University and Fellow of Gonville and Caius College, Cambridge, now Hardinge Professor of Mathematics in the University of Calcutta. And foremost among the mathematical productions in that University are: (1) A. R. Forsyth's Lectures introductory to the Theory of Functions of two complex Variables and (2) the remarkable work of which a review is attempted here. That work is an amplification of a course of lectures given for the University of Calcutta in the winter session of 1909-10. The student of mathematics comes across, frequently, in modern analysis, the term "matrix" and the fundamental laws of a matrix theory, but no-where does he find, search however he may, a *complete* and *consistent* calculus of matrices. "To give a satisfactory answer then to the frequently propounded question 'what is a matrix?'; it seemed advisible to Dr. Cullis to commence with some account of the Theory"; the result was that a decent volume of 430 pages was published in 1913, giving the most fundamental portions of the theory and concluding with the solution of any system of linear algebraic equations which is treated as a special case of a matrix equation of the first degree. The contents of the first volume can be broadly divided into two parts, *viz.*, the Theory of Determinoids and the Theory of Matrices. The theory of determinants has hitherto held the sway and Sir Thomas Muir, the learned author of the two well-known volumes on the History of that Theory, has expressed a doubt (vide Mathematical Gazette for December, 1913) whether the "Theory of Determinoid" as developed by Dr. Cullis will contribute to the advancement of science to the same extent as its fellow-subject Matrices or its prototype Determinants. But the "determinoid" stands in the same relation to the *rectangular* matrix as the "determinant" stands to the *square* matrix and surely, it is incumbent upon a mathematician to give, a scientific treatment, of the *most general* case of the *rectangular* matrix and its *content*, the determinoid. The aim of the author has been to construct and

develop the theory with a view to its extensive *applications* in Algebra, Geometry, Applied Mathematics, Vector Analysis and the Theory of Invariants. But the theory alone has been so engrossing in its character that the *applications* have had to be put further back. The author need however make no apology for, nor the mathematical world should mind, the delay, We should recollect Dr. Hobson's correct summing-up of the position of a true mathematician and the appraisement of his work. "Any attempt," says he, "to discourage perfectly untrammelled research in those parts of the subject that are most remote from *practical* interests, or that show least promise of fruitful *application* in other branches of science, would not only be a *vital* blow to mathematics as an evergrowing science, but would ultimately impair its efficiency as an *instrument*. Mathematics can only flourish if it has full *autonomy*. The nature and direction of its future applications in pure and applied science can never be fully foreseen. It is, however, extremely probable that the services it will render in the future will cover an even greater range than in the past, provided it is allowed to fit itself for rendering such services by according it full opportunity to develop itself in accordance with its own nature."

In taking stock of a recent work (Dickson's History of the Theory of Numbers), Professor Lehmer very appropriately says, "In these days when 'pure' science is looked upon with impatience, or at best with good-humored indulgence the appearance of such a book will be greeted with joy by those of us who still believe in mathematics for mathematics' sake." The reviewer is in good company, when he expresses his firm faith that "no great headway will ever be made in any science, least of all in Mathematics, by those who are always looking for the penny." He is also firm in the faith that as the years go by, the Matrix Theory (like the Theory of Numbers) will become increasingly popular, and works like Dr. Cullis's will contribute in a large measure, to render that theory eminently attractive. The reviewer would proceed now to the contents of that work.

The Determinoid.

The first volume consists of a preface, eleven chapters and a complete and useful index. Each chapter is proceded by a summary of its contents—a new feature. It contains "the *foundations* of a Calculus of Matrices in which the oparations are, addition, subtraction

or multiplication and the result of performing any number of these operations with any rectangular matrices whatever is always a completely determinate matrix. It also contains an account of the properties of the *determinoid* of a rectangular matrix which becomes the *determinant* of the matrix in the particular case when the matrix is square." The book is so remarkable, so elaborataly written, so logical in its methods and so much new ground has been broken that the reviewer does not feel that he needs offer an apology for entering into the *details* of each chapter and indicating some of the results arrived at.

Chapter I gives an introductory account of *rectangular* matrices, of determinoids and a description of various abbreviated notations, both for matrices and determinoids, used in the text. Dr. Cullis writes the rectangular matrix of m horizontal and n vertical, rows (notice that he drops the time-honoured term "column" and rightly, I think) as

$$A=[a]^{n}_{m}=\begin{vmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ a_{m1} & a_{m2} \quad a_{mn} \end{vmatrix}$$

and the determinoid of the matrix as

$$(a)^{n}_{m}=\begin{vmatrix} a_{11}\ldots a_{1n} \\ a_{12}\ldots a_{2n} \\ a_{m1}\ldots a_{mn} \end{vmatrix}=\det\,[a]^{n}_{m}.$$

Other matrices can obviously be derived from the parent matrix A by—

(*i*) A rearrangement, in *any manner*, of the vertical and horizontal *rows* among themselves without any row being entirely struck out. This is called a *derangement* of A.

(*ii*) Striking out some rows of one kind or of both kinds, leaving the others in the *same relative orders* as in the matrix A. The matrix thus left is called a *corranged minor* matrix of A. When such a matrix is derived by the striking out of only *one kind* of rows it is called a *simple* minor matrix.

And (*iii*) by striking out rows of one kind or of both kinds from a derangement of A. The residue is called a *deranged* minor matrix of A.

The *corranged* and *deranged* minor matrices and the derangements, of A constitute, in their totality, the *derived* matrices of A. A matrix is said to *contain* each of its minor matrices, deranged and corranged, thus giving significance to the "root" meaning of the word "matrix" and illustrating to some extent the remark of Sylvester (one of the great pioneers in the theory of matrices) that "a matrix is a rectangular array of terms, out of which different systems of determinants (determinoids) may be engendered as from the *womb* of *a common parent*; these cognate determinants (determinoids) being by no means isolated in their relations to one another but subject to certain simple laws of mutual dependence and simultaneous deperition." The object of the theory of matrices is to exhibit these laws and their consequences.

From the definition of derived matrices, the author passes on to the definition of *complete* and *incomlete* derived products of the *elements* of the parent matrix (Sub Art. 3) and to the definition of a *step* (backward or forward, horizontal or vertical) (Sub Art. 4), thus paving the way for the definition of a *determinoid* (Art. 3), *viz.*, that the determinoid of any matrix A, is the *algebraic* sum of *all* its *complete* derived products (a complete derived product being the product of the *largest possible* number of elements selected so that no two of them occur in the same vertical or horizontal row) when each product is subject to a *rule of sign*, the elements of the parent matrix being assumed, for present purposes, to be *scalar* numbers, and the *order* of the factors being immaterial. Dr. Cullis here introduces the term "affect" of a complete product which determines its sign. Thus, if $, β', γ', δ',... are the *sums* of the horizontal and vertical steps by which, counting from the leading element, we arrive *respectively* at each element α, β, γ, δ,... of a complete product $\alpha\beta\gamma\delta$...beginning with the original matrix and then passing on successively to the matrices obtained by striking out in turn the vertical and horizontal rows which cross at any particular element, the number $\omega=\alpha'+\beta'+\gamma'+\delta'+\ldots$ is called the *affect* of the product $\alpha\beta\gamma\delta\ldots$ and the sign to be attached to the latter is determined by $(-1)^{\omega}$. The question arises, is this rule of sign *self-consistent*? Dr. Cullis gives a complete proof, in small type, that the answer is in the affirmative. The determinoid and, it is proved afterwards, in Art. 13, page 48, the algebraic sum of all the affected derived products of order r when r is not greater than the efficiency of the matrix are

Affect of a product.

thus shown to be *definite* functions of the elements of the matrix. The ground is now prepared for giving some elementary properties of determinoids. They are—

Properties of a determinoid.

(*i*) The value of a determinoid is unaltered by interchanging its horizontal and vertical rows. Thus a determinoid can always be written with its long rows horizontal and two conjugate matrices (defined in Art. 5 but it appears to the reviewer that it might have been defined earlier, say, just after Art. 1 giving the definition of matrices of various orders) have their determinoids equal in value, although they, the matrices themselves, are generally different.

(*ii*) A determinoid is a homogeneous *linear* function of the elements of any *long* row.

(*iii*) If each element of a *long* row of a determinoid is split up into the sum or difference of two other elements, the determinoid is split up into the sum or difference of two other determinoids. This can be easily generalized.

(*iv*) Multiplying every element of any long row by a number κ is equivalent to multiplying the determinoid by κ.

(*v*) If any two *long* rows are interchanged, the determinoid is changed in sign but is unaltered in magnitude. Thus, when the elements of two long rows are identical, the determinoid vanishes and the value of a determinoid is unaltered by adding to the elements of any *long* row the corresponding elements of any other long row, each multiplied by the same number. (Dr. Cullis uses in Sub Art. 5 of Art. 5, page 20, the word "quantity" which should more appropriately I think be "number").

Chapter I is concluded with the pertinent remark that "as regards *long* rows, the properties of determinoids are closely analogous to those of determinants which are of course, a special class of determinoids."

Chapter II deals with the *affects* of the elements and derived products of a matrix or determinoid. At the outset it is discovered that the *affects* of the *elements* of a matrix have not been formally defined. This ommission has been made good in small type in the preface to the first volume as Art. 5a. The *affect of the element* $a_{x,y}$ of the matrix $[a]_m^n$ is defined to be the number ω given by the equation $\omega=(x-1)+(y-1)$. $(x-1)$ is called the vertical affect and $(y-1)$, the horizontal

Affects of the elements of a matrix.

affect of $a_{x,y}$ the total affect or simply, the affect of $a_{x,y}$ being their sum. This being premised it is natural to define the affect, ω, of a derived product of order r, *viz.*, $P=z_1z_2\ldots z_k\ldots z_r$, to be $\omega_1+\omega_2+\ldots\omega_k+\ldots\omega_r$, where ω_k, is the affect of the element z_k in matrix A_{k-1} obtained from the original matrix A by the *successive* deletion of $k-1$ vertical and horizontal rows passing through the previous $k-1$ elements, $z_1, z_2,\ldots z_{k-1}$. This is, in substance, the rule of signs given in Art. 3 of Chapter I when applied to any derived product of a matrix or determinoid instead of only to the *complete* derived products of the latter. The splitting-up of ω_k into two elements ω_k' (the vertical affect) and ω_k'' (the horizontal affect) is obvious and we get $\omega=\omega'+\omega''$ where $\omega'=\Sigma\omega_k'$ and $\omega''=\Sigma\omega_k''$. Clearly, in practice, the determination of affects of derived products by the striking out of rows and the counting of steps is inconvenient and practical rules are required for simplifying the process. This desideratum is attained in Sub. Art. 3 of Art. 8 in which rules are given which can be worked by the mere inspection of the relative *positions* of the elements with respect to horizontal and vertical rows of the matrix and in the product itself. In Sub. Art. 4 of Art. 8, the important fact is deduced that the affect of a derived product when it is *extended* or *completed*, (extension and completion having been defined in Art. 6) is not changed; whence it follows that the affects of all derived products can be reduced to the determination of the affects of complete derived products. Sub. Art. 2 of Art. 8 gives simple rules for doing this. In reading this sub-article, however, it seems to the reviewer, that the expression "horizontal affects" in line 5 of the second paragraph should be "vertical affects" and that the expression "vertical affects" in line 6 should be "horizontal affects." Similar changes would also be necessary in the sentence of this paragraph which follows "vertical affects" in line 6. Further down also, at the bottom of the page, (Ex. iv) the expression "vertical suffixes" should be replaced by "horizontal suffixes."

Changes in the affect of a derived product caused by the interchange of two *consecutive* parallel rows and, more generally, of *any* two parallel rows of a matrix, are exhaustively dealt with in Arts. 9 and 11 of this Chapter and copiously illustrated.

Changes in the affect of a derived product by the interchange, in the *product*, first of *consecutive* suffixes of the same kind, and, secondly, of *any* two suffixes of the same kind, are similarly dealt with in Arts. 10 and 12. Art. 13 brings prominently to notice

(as already referred to above) the *invariance* of the *sign* of a derived product—complete or incomplete. We are now ready for an important operation in the theory, *viz.*, the reduction of any derived product of a matrix M to a *leading product* (the reviewer misses the definition of a "leading product") by forward moves, the word "move" with the qualifications "vertical and horizontal," "forward" and "backward" being defined (Art. 14). In the illustrative example (*iii*) below the Article, it is logically deduced that "the *determinoid* of a square matrix," as developed in Dr. Cullis's Calculus, is identica with the *determinant* of that matrix, as ordinarily defined.

We have hitherto been concerned with changes, so to speak, due to the motion of *translation* of entire rows. Art. 15 introduces changes due to the motion of *rotation* through two right angles in the rectangular array called the matrix, round three mutually perpendicular axes, two lying in the plane of the rectangle bisecting the opposite sides and the third perpendicular to that plane, through its centre. The results obtained are important for future use and, for immediate use in Art. 16, which gives the results of the three kinds of *inversion* of the orders of arrangement of the rows of a determinoid, $(a)_m^n = \Delta$ They are as follows :—

I. When the *long* rows are inverted :—

$$\Delta' = (-1)^{\frac{m(m-1)}{2}} \times \Delta.$$

II. When the *short* rows are inverted :—

$$\Delta' = (-1)^{m(n-m)+\frac{m(m-1)}{2}} \times \Delta.$$

III. When *both* sets of rows are inverted :—

$$\Delta' = (-1)^{m(n-m)} \times \Delta.$$

Sequences and their affects.

Chapter III dealing with "*sequences* and the affects of derived *sequences*" may well be characterized as a 'simplifying' chapter and opinion may be divided as to whether it should not have found an earlier place in the work. A *sequence* is any *linear* arrangement of *elements* which are letters or numbers, such as :—

$$A = [a_1 a_2 \ldots a_n].$$

It can, of course, be regarded as a matrix with only one long row and it is possible to adapt the definitions and some of the deductions in regard to the matrix $[a]^{*}_{m}$ in the previous chapters, *mutatis mutandis* to *sequences*. But Dr. Cullis with his logical acumen, works out a new chapter, an interesting one in Combinatory Analysis. Specially valuable are the results obtained, in Art. 19, concerning the affects of derived sequences which are of use in the following chapters, in view of the remarkable result obtained at the end of the chapter, in Art. 22, *viz.*, the reduction of the affects of *derived products* (of a matrix) to the affects of *sequences*. Thus, "all properties of the *affects* of derived products of a matrix or determinoid, can be deduced from the properties of affects of *sequences*." This is, indeed, a great simplification.

Having dealt successively with the affects of *derived* products of a fundamental matrix or determinoid and the affects of *derived* sequences of a fundmental sequence, the author naturally passes on, in Chapter IV, to affects of derived matrices and derived determinoids. Obviously a definition of the *affect* of a derived matrix (which would be applicable to a derived determinoid also), is needed and this is given in Art. 23 in the most natural manner. If $A=[a]^{*}_{m}$ be the fundamental matrix and $B=[a_{x_y}]^{n}_{v}$ be any *derived* matrix (page 3) of A, then, writing B in the expanded from in double-suffix notation as

$$\begin{vmatrix} a_{x_1 y_1} & a_{x_1 y_2} \dots & a_{x_1 y_r} \\ a_{x_2 y_1} & a_{x_2 y_2} & a_{x_2 y_r} \\ \dots & \dots & \dots \\ a_{x_v y_1} \dots & \dots & a_{x_v y_r} \end{vmatrix}$$

the affect of the derived matrix B in the fundamental matrix A is *defined* to be the quantity ω given by the equations :—

ω'=the affect of the *sequence* $[x_1 x_2 \dots x_v]$ in the *sequence* $[1\ 2 \dots m]$ and is called the vertical affect of B in A; ω'=affect of the *sequence* $[y_1 y_2 \dots y_v]$ in the *sequence* $[1\ 2 \dots n]$ and is called the horizontal affect of B in A.

$$\omega = \omega' + \omega''.$$

But, if the definition is to be useful, it must make us independent of any *particular* notation used in writing the derived matrix,

Dr. Cullis therefore gives us two more differently worded definition on page 88.

As in the case of derived products and derived sequences, so also in the case of derived matrices or d^{e}terminoids, rules are given for the *extension* and *completion* of matrices derived from a fundamental matrix and it is shown that the *affect* of a derived matrix is not altered by *extension* or *completion* (Art. 24). Art. 25 is concerned with several important theorems relating to the *affects* of derived matrices which are, naturally, analogous to the theorems in Art. 19 relating to the affects of derived sequences. In Art. 26 *complementary* derived matrices are defined and the two theorems VIIIa and VIIIb of Art. 19 regarding complementary *sequences* are extended to complementary *matrices* or *determinoids*.

Epaxnsion of a determinoid.

The reader is now fully equipped for dealing with the *expansion of a determinoid* which is dealt with in Chapter V. Three methods are first given of such expansion, *viz.*—

(1) Expansion in terms of the elements of any *given* long row (Art. 27).

(2) Expansion in terms of the simple minor determinants (Art. 30).

And (3) expansion in terms of the simple minor *determinants* of a given *long cut* minor as defined in Art. 31 (Art. 32).

Art. 28 gives only a glimpse of reciprocal matrices and determinoids and the three examples appended to the article in small type give some relations which are obvious consequences of the definitions and the first method of expansion.

Properties of short rows.

Art. 29 deals with the properties of the *short rows* of a determinoid. It is shown that the determinoid is in general, a *non-homogeneous* linear function of the elements of any selected *short* row.

It is also shown that in the case of the determinoid $(a)_{m}^{n}$ where n is $>m$:

(*i*) we can expand the determinoid in terms of the elements of any given $n-m+1$ short rows,

(*ii*) the value of a determinoid is not altered by *post-fixing* an *additional* short row of zeros *after* the existing short rows,

and (*iii*) the value is multiplied by $(-1)^{n}$ by *prefixing* such an additional short row.

All these results are consequences of the general theory of Art 30 in which the expansion of a determinoid in terms of its *simple minor determinants* is effected. This is noticed below :—

Let A be any determinoid with m long rows and n short rows. Let $D_1\ D_2 \ldots D_i \ldots D_\nu$ be the *determinoids* which can be formed from A by the omissions of short rows. These are the *corranged* simple minors of A which are determinants. Their number is $\nu = \binom{n}{m}$. Then, it is easily established,

$$A = (-1)^{\omega_1} D_1 + (-1)^{\omega_2} D_2 + \ldots (-1)^{\omega_\nu} D_\nu.$$

where ω_i is the affect of D_i in A.

This expansion, it is then proved, remains true when the simple minor determinants are *deranged* any way. So that "A determinoid is the algebraical sum of all *its* distinct simple minor determinants, *corranged* or *deranged*, when each of these determinants has the sign determined by its affects in the original determinoid." This is an important result.

Art. 31 gives, (*i*) the classification of simple minor *determinoids* into long-cut and short-cut minors (*ii*) the definition of the *reduced order* of the simple minor determinoid and (*iii*) the definition of of *superior* and *inferior* minors.

These definitions might, as they cover matrices also, have come earlier say, in Chapter I but Dr. Cullis has obviously a plan of his own according to which he succeeds in making each chapter self-contained.

Generalization of Laplace's development of a determinant.

Art. 32 gives the expansion of a determinoid in terms of the simple minor determinants of a given *long-cut* minor matrix of the matrix of the derterminoid Setting

$$\Delta = [a]^{n}_{m} = \det [a]^{n}_{m}$$

$$U = [a_{\nu 1}]^{n}_{\nu}$$

$B = (b)^{n}_{n}$ = a corranged determinant of order n formed from n by striking out $n - u$ *short* rows.

$C=(c)_{\rho}$=a corranged determinoid complementary to B in Δ.

ω=affect of B in Δ.

We obtain

$$\rho=m-u \quad \sigma=n-u$$

$$\Delta=\Sigma(-1)^{\omega} BC.$$

The number of terms in the sum is obviously $\binom{n}{u}$.

The proof given is lucid and logical and defining the cofactor of any minor determinant B (corranged or deranged) of a determinoid (or matrix) Δ, to be the *corranged* minor determinoid *complementary* to B in Δ and affected with the sign determined by the affect of B in Δ, Dr. Cullis arrives at what may be regarded as a *generalization* of Laplace's deveolopment of a determinant, *viz.*, "A determinoid is the algebraical sum of all the products which are obtained when every distinct simple minor determinant of a given *long-cut* minor matrix is multiplied by its cofactor." Illustrative examples follow and the student of the calculus of determinoids will do well to remember the interesting result obtained in exapmple V (page 120), *viz.*, if Δ is the determinoid

$$\begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{vmatrix}$$

then $\Delta=0$ or $(a)_m$ *i.e.*, the determinoid of $[a]_n$, according as $n-m$ is even or odd.

Below the examples on page 120 is given an important *theorem*, which is not given in bold black type as other theorems are. Moreover, the theorem concerns *similar* matrices of which no mention is made in previous pages. It is not till we come to Art. 39, page 153 that we find *similar* matrices defined.

Art. 33 gives a striking theorem involving an important function. The theorem is $S_s = \Delta \times \underset{s-m}{\overset{s-m}{Q}}$, where S_s is the algebraical sum of all the corranged superior simple minor determinoids of reduced order s derived from a fundamental determinoid Δ with m long rows and n short rows, when each minor has its sign determined by its affect in Δ and $\underset{s-m}{\overset{s-m}{Q}} = \Sigma(-1)_k$, the values of k being the *affects* of all possible corranged minor sequences of $n-m$ elements derived from a fundamental sequence of $n-m$ elements. The function, then, of importance is $\underset{m}{\overset{n}{Q}}$ and various properties of this are discussed in Art. 34. This is printed in small type, probably, because it is somewhat of a digression from the main topic under discussion. This function had, it may be noted, been previously discussed by Metzler in Vol. XXII of the American Journal of Mathematics as $\phi(n,m)$ and it is so interesting that we may notice some of the properties worked out by Dr. Cullis :—

In sub-article 2 (*ii*) of Art. 34 it is proved that

$$\underset{m}{\overset{n}{Q}} = \underset{n-m}{\overset{n}{Q}} \; ; \; m \not> n.$$

Sub-article (3) gives formulae of *reduction* for $\underset{m}{\overset{n}{Q}}$. They are

$$\underset{m}{\overset{m+r}{Q}} = \underset{m-1}{\overset{m+r-1}{Q}} + (-1)^{m} \underset{m-1}{\overset{m+r-2}{Q}} + (-1)^{2m} \underset{m-1}{\overset{m+r-3}{Q}}$$

$$\ldots + (-1)^{rm} \underset{m-1}{\overset{m-1}{Q}}$$

$$= \underset{0}{\overset{r-1}{Q}} + (-1)^{1} \underset{1}{\overset{r}{Q}} + \ldots (-1)^{m} \underset{m}{\overset{m+r-1}{Q}}.$$

With the help of these formulae of reduction a tabular representation of the values of the function for different values of m and n is given in sub-article 4. And in sub-article (5) it is shown that

$$\underset{2m}{\overset{2n}{Q}} = \underset{2m+1}{\overset{2n+1}{Q}} = \underset{2m}{\overset{2n+1}{Q}} = \binom{n}{m}$$

and that

$$\underset{2m+1}{\overset{2n}{Q}} = 0.$$

After discussing in Art. 35 and 36 algebraic sums of (inferior) short-cut and (inferior) long-cut simple minor determinoids of given reduced order, obtained from the fundamental determinoid, the author gives in Arts. 37 and 38 theorems which *generalize* the preceding results. These two articles contain an investigation of the algebraical sum of the products obtained by multiplying each affected simple minor determinoid of given reduced order, of a given simple minor matrix of the fundamental determinoid, by its *corranged* complement or cofactor. In case I of Art. 38, *i.e.*, when both the minor determinoid of given reduced order, A^{r}_{n} and its cofactor, A^{n-r}_{m-n} are *superior* simple minors of the respective matrices, the sum is a certain *numerical multiple, viz.*, $\pm Q^{m-n}_{n-r}$ of the fundamental determinoid. This clearly includes the result of Art. 32 and Dr. Cullis is justified in calling the result "*a still further generelization of Laplace's development of a determinant.*" Chapter V concludes with seven pages of closely written small type giving the formal proofs of the three cases of Art. 38, the concerete examples, illustrating these three cases and just preceding the proofs, being specially valuable, as leading to a right understanding of the proofs which are not easy to follow.

The Matrix.[1]

PHILONOUS—I am not for imposing any sense on your words: you are at liberty to explain them as you please. Only I beseech you make me understand something by them.

BERKELEY,
The First dialogue between Hylas and Philonous.

With Chapter VI, the theory *proper* of matrices begins. Hitherto the matrix has been considered only as a rectangular *arrangement* of the scalar numbers and the determinoid only has played a considerable part. Square matrices, unlike "determinoids" has a previous history which is summed up below, in the inimitable style of Sylvester

[1] Sir Thomas Muir gave in Vol 20 of the American Journal of Mathematics, a list of writings on the Theory of Matrices from 1857 to 1893. The Bibliography at the end of Shaw's Synopsis of Linear Associative Algebra brings the list down to 1905.

to whom and to Cayley, in its early stages, the theory of matrices owes so much for its development:—

A historical retrospect.

"A matrix of a *quadrate* form *historically* takes its rise in the notion of a linear substitution upon a system of variables or carriers; regarded apart from the *determinant* which it may be and at one time was almost exclusively used to represent, it becomes an empty schema of operation, but in conformity with Hegel's principle that the Negative is the course through which thought arrives at another and a fuller positive, only for a moment loses the attribute of *quantity*, if it be allowed that that term is properly applied to whatever is the subject of *functional operation*, of a *higher* and *unthought of kind*, and so to say, in a glorious shape—as an organism composed of discrete parts but having an essential and undivisible unity as a whole of its own. *Naturam expellas furca tamen usque recurret.* The conception of Multiple quantity thus rises upon the field of vision."

"At first undifferentiated from their content, matrices come to be regarded as susceptible of being multiplied together; the word multiplication strictly applicable at that stage of evolution to the *content* alone, getting transferred by a fortunate confusion of language to the schema and superseding, to some extent, the use of the more appropriate word, composition, applied to the reiteration of substitution in the Theory of Numbers. Thus, there comes into view a process of multiplication which the mind, almost at a glance, is able to recognize, must be subject to the Associative law of ordinary multiplication, although not so to the Commutative law; but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema and stood revealed as *bonâ fide* multiple Quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity. This revolution [1] was effected by a forcible injection into the subject of the concept of *addition*, *i.e.*, by choosing to regard matrices as susceptible of being *added* to one another; a notion as it seems to me quite foreign to the idea of substitution, the *Nidus* in which that of

[1] The late Maxime Böcher said "The Matrix first appears as *inanimate* object, a mere rectangular *array* of quantities from which determinants whose vanishing and non-vanishing are observed or whose values are used are cut out. Later, however, the matrix appears as an *animated being* capable of combination by *addition* and at least two kinds of multiplication with its kind and it is in *this aspect that the conception becomes a highly fruitful one.*"

multiple quantity was laid, hatched and reared. This step was as far as I know, first made by Cayley in his memoir on matrices (1858, Vol. II, Collected Works) wherein he may be said to have laid the foundation-stone of the science of Multiple Quantity (*N. B.*—But Willard Gibbs contended, and rightly I think, that the key to the theory of matrices is given in the first Ausdehnungslehre, *i.e.*, of 1844 ; *vide* page 167, Vol. II, of the scientific papers of Gibbs). That memoir indeed (it seems to me) may, with truth, be affirmed to have ushered in the reign of Algebra the Second ; just as Algebra the First, in its character, not as mere art or mystery, but as science and philosophy took its rise in Harriot's *Artis Analyticae Praxis*, published in 1631, ten years after his death, and exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir, the idea of subjecting matrices to the *additive* process and of their consequent amenability to the laws of functional operation was not taken from it but occurred to me independently before I had seen the memoir or was acquainted with its contents ; and indeed forced itself on my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices. My memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was *the indirect cause of turning my attention* to the subject. All this only proves how far the discovery of the quantitative nature of matrices is removed from being *artificial* or *factitious*, but, on the contrary, was bound to be evolved, in the fulness of time, as a necessary sequence to previously acquired cognitions."

" Already in Quaternions the example had been given of Algebra released from the yoke of the commutative principle of multiplication —an emancipation somewhat akin to Lobatchewsky's of Geometry from Euclid's noted empirical axiom, and later on the Pierces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the Associative law of multiplication, would be eventually found to be identical with linear transformations of schemata susceptible of matricular representation."

" That such must be the case it would be rash to assert ; but it is very difficult to conceive how the contrary can be true or where to seek, outside the concept, of *substitution* for matter affording pabulum

to the principle of free consosciation of successive actions or operations."

Cayley in his "Memoir on the Theory of Matrices" (Vol. II of his Collected Mathematical Works—pages 475-496) writes the matrix of the third order as

$$\begin{pmatrix} a & b & c \\ ' & b' & c' \end{pmatrix}$$

"The notion of such a matrix" he says "arises naturally from an abbreviated notation for a set of *linear* equations, *viz.*, the equations

$$X = ax + by + cz.$$
$$Y = a'x + b'y + c'z.$$
$$Z = a''x + b''y + c''z,$$

may be more simply represented by

$$(X, Y, Z) = \begin{pmatrix} a, & b, & c \\ ' & b' & c' \\ '' & b'' & c'' \end{pmatrix} (x, y, z)$$

and the consideration of such a system of equations leads to most of the fundamental notions in the theory of matrices."

Writing the equations

$$(X, Y, Z) = \begin{pmatrix} a, & b, & c \\ a'' & b'' & \end{pmatrix} (x, y, z)$$

and

$$(x, y, z) = \begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{pmatrix} (\xi, \eta, \zeta)$$

Cayley obtains

$$(X, Y, Z) = \begin{pmatrix} A, & B, & C \\ A' & B' & C'' \\ A'' & B'' & C''' \end{pmatrix} (\xi, \eta, \zeta)$$

$$\begin{pmatrix} a, & b, & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \begin{pmatrix} a, & \beta, & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{pmatrix} (\xi, \eta,$$

$$= \begin{pmatrix} a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \begin{pmatrix} a' & \beta' & \gamma' \\ a'' & \beta'' & \gamma'' \end{pmatrix}$$

From which substituting for the matrix

$$\begin{vmatrix} A, & B, & C) \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}$$

its value, obtains the well known rule for the multiplication or composition of two matrices.

In his celebrated memoir Cayley restricts his investigation mainly to *square* matrices. It is shown that matrices (attending to those only of the same order) "comport themselves as single quantities; they may be *added, multiplied* or compounded together, *etc.* The law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition) there is the peculiarity that matrices are not in general *convertible*; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix and thence to arrive at the notion of a rational and integral function or generally of any algebraic function, of a matrix. Cayley obtains the remarkable theorem that *any matrix whatever satisfies an algebraic equation of its own order*, the co-efficient of the highest power being unity, those of the other powers being functions of the terms of the matrix, the last co-efficient being in fact the determinant. This theorem shows that every rational and integral function (or indeed every rational functional) of a matrix may be considered as a rational and integral function, the degree of which is at most equal to that of the matrix less unity; it even shows that in a sense the same is true of any algebraic function whatever of a matrix. One of the applications of the theorem is the *finding of the general expression of matrices which are convertible with a given matrix*. The theory of rectangular matrices appeared to Cayley much less important than that of a square matrices and he devotes only a few articles to them at the end of his memoir.

As the very general theorem referred to above formed the basis of subsequent developments of the Matrix Theory it may he interesting to mention it here :—

The Identical Equation. Let M be any matrix $[a]_m^m$. Then the derived determinant

$$\begin{vmatrix} a_{11}-M, & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22}-M & \dots & a_{2m} \\ \dots & \dots & & \dots \\ \dots & \dots & & \dots \\ a_{m1} & \dots & & a_{mm}-M \end{vmatrix}$$

vanishes considered as a *matrix* equation of the m-th order. Cayley writes the Identical equation, symbolically, as Det $(\overline{1}.\ M-\overline{M}.\ 1)=0$ where the matrix M considered as a single entity (quantity) is represented by $\overline{M}$ and taking 1 to denote the matrix unity, $\overline{1}.\ M$ would represent the matrix M and M.1 would represent the matrix M considered as a single quantity *involving the matrix unity*.

Cayley gives the proof of the Identical equation in the case of the matrix of the second order and gives only the enunciation in the case of the third order. An elementary proof in the case of the matrix of the third order (which is applicable also to the general case of any order) is given by Forsyth in the Messenger of Mathematics, Vol. 13 as well as by Muir in his article on Bipartite Functions and Determinants in the Proceedings of the London Mathematical Society, Vol. 16. Arthur Bucheim's elegant paper on the "Theory of Matrices" (Vol. 16 of the Proceedings of the London Mathematical Society) connects Grassmann's methods (in which the key to the matrix theory lay) with the theory created by Cayley and Sylvester, by connecting them both with Hamilton's investigations. Bucheim shows that the linear and vector function of a vector is simply the matrix of the third order.

A brief survey, which is here attempted of the previous history of the Matrix Theory some account of which is necessary in order that we may be able to make an estimate of the remarkably original work of Dr. Cullis, will be incomplete if we did not refer to the writings of W. H. Metzler and Henry Taber. Metzler wrote on "the roots of matrices" (Vol XIV of the American Journal) and on "certain properties of symmetric, skew-symmetric and orthogonal matrices (Vol XV, *ibid*). His paper on the function $\phi\ (n, m)$ has been already noticed above. Taber's most important paper on the theory of matrices is that contained in Vol. 12 of the American Journal of Mathematics. According to him "Hamilton must be regarded as the originator of the theory of matrices, as he was the first to show that the symbol of a linear transformation might be made the subject matter of a calculus. Cayley makes no reference to Hamilton (in his Memoir of 1858) and was (probably) unaware that results essentially identical with some of his, had been obtained by Hamilton (or was it because Quaternions did not appeal to him as it did not to Lord Kelvin !!). On the other hand, Hamilton's results related only to matrices of the third and fourth order while Cayley's method was absolutely general.

The identity of the two theories was explicitly mentioned by Clifford in a passage of his Dynamic (Clifford's writings on the subject are contained in Vols. I, II and III of his Dynamic, page 162 etseqelae and also in his Collected Papers, pp. 337-341).

Linear Associative algebras.

Subsequent to Cayley but previous to Sylvester, the Pierces, especially Charles Pierce, were led to the consideration of matrices from a *different point of view ; viz.*, from the investigation of Linear Associative Algebras, involving any number of linearly *independent units*. In this aspect the quantities arranged in a square are looked upon as the scalar co-efficients of the several units or "vids" of an algebra belonging to a certain class. Charles Pierce did something more. In his remarkable investigations upon the extension of the Boolian Calculus (*vide* Boole's Laws of Thought recently reprinted by the Open Court Publishing Company) to the Logic of relatives (*vide* American Journal of Mathematics, Vol. IV, page 221 and the memoirs of the American Academy of Sciences for 1870 and 1875) he came upon a set of forms (the "vids" mentioned above), constituting a system virtually identical with the calculus of matrices. He made the discovery that the whole theory of Linear Associative Algebra is included in the Theory of matrices ; in particular, he gave the matrix form of all the algebras considered by his father Benjamin Pierce, in his Linear Associative Algebra (American Journal, Vol. IV). A simple example may be given here to show how a *matric* algebra may be viewed as a *linear* algebra :—

Consider, to fix ideas, the two matrices of the 2nd order

$$m=\begin{bmatrix} ab \\ cd \end{bmatrix} \quad \mu=\begin{bmatrix} a\beta \\ \gamma\delta \end{bmatrix}$$

and the four matrices

$$e_{11}=\begin{bmatrix} 10 \\ 00 \end{bmatrix} \quad e_{12}=\begin{bmatrix} 01 \\ 00 \end{bmatrix}$$

$$e_{21}=\begin{bmatrix} 00 \\ 10 \end{bmatrix} \quad e_{22}=\begin{bmatrix} 00 \\ 01 \end{bmatrix}$$

The sixteen products of these four by two's are

$$\left.\begin{aligned} e_{ij}e_{jk}&=e_{ik} \\ e_{ij}e_{tk}&=0(t\neq j) \end{aligned}\right\} \quad \ldots \quad (A)$$

Then clearly

$$
\begin{aligned}
m &= ae_{11} + be_{12} + ce_{21} + de_{22} \\
\mu &= \alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22} \\
m+\mu &= (a+\alpha)e_{11} + \ldots (d+\delta)e_{22} \qquad \ldots \text{(B)} \\
m\mu &= (a\alpha + b\gamma)e_{11} + (a\beta + b\delta)e_{12} \\
&\quad + (c\alpha + d\gamma)e_{21} \\
&\quad + (c\beta + d\delta)e_{22}.
\end{aligned}
$$

The set of hyper-complex numbers $ae_{11} + \ldots + de_{22}$ in which $a \ldots d$ range independently over a domain of rationality Ω and for which addition and multiplication are defined by (B), is called a linear associative algebra over the same domain with the four units e_{11}, e_{12}, e_{21}, e_{22} subject to the multiplication Table (A).

The above can be easily generalized, when m and μ are matrices of higher order than 2.

To Sylvester, in the 19th century is due most of the development in the theory. In his lectures on universal Algebra "he distinguished between the different degree of *vacuity* and *nullity* of a matrix, replacing by these terms, the term "indeterminate" used rather vaguely, by Cayley to denote a "null" or "vacuous" matrix. Sylvester showed also how to derive a *chain of equations* (the *catena*) from the *Identical Equation* mentioned above and the relation of these to the Latent function of two or more matrices taken in a certain order. The terms "nullity," "vacuity" and "latent function" are of such importance in Sylvester's Theory that a brief description may be given of them.

Let ϕ be the matrix $[a]_{w}^{w}$ of order w.

If all the minors of the matrix of order $w-i+1$ vanish, but not all the minors of order $w-i$, its nullity is of the order i. If $i=1$, it is said to have "simple nullity." The term "vacuity" has reference to the number of roots, *equal to zero* of the equation

$$
\begin{vmatrix}
a_{11}-\lambda & a_{12} \ldots & a_{1w} \\
a_{21} & a_{22}-\lambda & a_{2w} \\
a_{w1} & \ldots \quad \ldots & a_{ww}-\lambda
\end{vmatrix} = 0
$$

If i roots of this equation are zero, the *vacuity* of ϕ is i; if $i=1$, ϕ is said to be "simply vacuous."

The roots of the above equation are termed by Sylvester the latent roots of ϕ and the determinant written above (when it does not vanish) is called the Latent function of ϕ and written as $|\ \phi-\lambda\ |$, λ being a scalar.

The "nullity" of ϕ cannot exceed its "vacuity" and since the latent roots of $\phi-\lambda$ are severally equal to the several latent roots of ϕ *less* λ, if g is an m-tuple latent root of ϕ the vacuity of $\phi-g$ is m.

Sylvester's Law of nullity is :—

"The nullity of the product of *two* (and, therefore, of any number of) matrices cannot be *less* than the nullity of any factor, nor *greater* than the *sum* of the nullities of the several factors which make up the Product."

When we see that the "nullity"[1] of a matrix, as defined by Sylvester, is the difference between its order and its "rank" as defined in Art. 68 of Dr. Cullis's work, Sylvester's law is equivalent to "If γ_1 and γ_2 are the ranks of two matrices of order n and R the rank of their product, then

$$R>\gamma_1+\gamma_2-n \qquad \ldots \quad (i)$$

$$\left.\begin{aligned} R&\leqq\gamma_1 \\ R&\leqq\gamma_2 \end{aligned}\right\} \qquad \ldots \quad (ii)$$

Dr. Cullis's Theorem III (generalized in Theorem IV) of Art. 71 gives only (*ii*) above, which is only *one half*[2] of Sylvester's famous Law which has been extended by Bucheim (Phil. Mag., Vol. 18, 1884). There is no doubt that the Identical Equation and its implications have received further developments in the skilful hands of Dr. Cullis in the subsequent volumes of this great work.

[1] Connected with a matrix of order n, are two integers, its rank and its nullity. The sum of these two is n, so that we may dispense with either one of them and most authors use the "rank" alone. It is however convenient to use the Nullity and it is the generalization of this latter conception and not of the rank, with which we have to deal in the case of Infinite and Fredholm determinants (Maxime Bocher).

[2] In Art. 133, page 177, Chapter XV of the second volume, the author has given the *complete* discussion of the rank of a matrix product and it is noteworthy that Sylvester's Law is only the simplest possible case of the general law.

Dr. Cullis's development of the Theory of matrices of which only the fundamentals are given in Chapter VI, is entirely on an *independent* plan. The *similarity* and *equality* (real and conventional) of two matrices, their *addition* and *subtraction* and the *multiplication* of a matrix by a scalar number, are defined and considered in Arts. 39-41 and fully illustrated by examples. The definition of "zero" matrices given in sub-article 4 of Art. 39 is to be carefully attended to. When we write $A=0$, we mean A is a matrix, *every one* of whose elements is zero,—our familiar scalar zero. With this exception, *a matrix cannot be equated to a scalar number*. And it is proved further on (page 179) that "there is no matrix which can be regarded as being always equivalent to the non-zero scalar number k for purposes of multiplication."

Addition and subtraction of two matrices.

In Arts. 42-46, a *product* of two matrices is defined and the chief properties of such a product are discussed. In Art. 42, *active* and *passive* rows of the factor matrices and the *activity* and *passivity* of a factor matrix are defined (cf. similar notions in Vol. IV of Sylvester's collected works, page 210). The Law of the formation of the product, $[c]_n^s$ of two matrices $[a]_m^r$ and $[b]_r^s$ of *equal passivity* r is defined by

Product of two matrices.

$$c_{i,j}=\Sigma_u a_{i,u}b_{u,j}\ldots(u=1, 2,\ldots r)$$

This definition is equivalent to the *two* following statements :—

(1) The *product* matrix has the *same number of horizontal rows* as the first factor matrix and the *same number* of *vertical rows* as the second factor matrix.

(2) The element $c_{i,j}$ common to the i^{th} horizontal row and the j^{th} vertical row of the product matrix, is the sum of the products obtained by multiplying each element of the i^{th} horizontal row of the first factor matrix by the corresponding element of the j^{th} vertical row of the second factor matrix.

When the passivities of the two factor matrices are *not equal*, sub-article 3 of Art. 42 puts clearly how the product is to be formed by (*i*) the *insertion* of *additional* final *passive* rows in the factor which has the smaller number of passive rows or (*ii*) by striking out the *redundant* final passive rows in the factor with the larger number

of passive rows, these two processes being shown to be equivalent. This leads to the generalized statement :—

$$[a]_m \; [b]_s = [c]_m$$

where $$c_{i,j} = \Sigma_u \; a_{i,u} \; b_{u,j}$$

u receiving all integral values from 1 to the *smaller* of the two numbers r and s.

After giving in Arts. 43 and 44 respectively, the properties of passive and active rows in a product of two matrices (which follow easily from the definition given above), the author proves in Art. 45 two well-known propositions in the Theory of matrices, *viz.*,

(*i*) The product of two matrices taken in *specified order* is distributive (sub-article 2)

(*ii*) The product is in *general* not commutative (sub-article 3). The Note appended to sub-article 3, however, gives two instances of *special* commutative products, *viz.*,

(1) The product of a square matrix and its conjugate reciprocal (sub-article 6 to Art. 46)

(2) The product of two square matrices of the same order when such product is a non-zero *scalar* matrix (example *i* of Art. 47).

On this subject of commutativity, the memoir of John Byrnie Shaw, the reputed author of a masterly "synopsis of linear Associative algebras" and of the "Philosophy of Mathematics," in Vol. 4 of the Transactions of the American Mathematical Society (pp. 251-287) on the Theory of Linear Associative Algebra may be usefully consulted.

Art. 46 deals with some *special* cases of a product of two matrices which are of frequent use in the Chapters which follow.

One noteworthy feature of the author's treatment of his subject is that he leads his reader on *step by step*, from particular cases to more general ones. After dealing exhaustively with the product of *two* matrices, he introduces the product of three or more matrices, very suitably calling that a *chain* of matrix factors, and showing how what has gone before is applicable to the chain. In particular, he proves (Art. 47) the Associative property of a product of three matrices and the next Article extends the proof to the product of any number of

Chain of matrix factors.

matrices. In these two articles, he also proves the *Distributive* property of any product.

In the case of a *chain* of matrix factors, it is found necessary, for obvious reasons, to redefine, *active* and *passive* rows, *activity* and *passivity* and the *efficiency* of the product (Art. 49).

One-to-one correspondence between the active rows of the two *extreme* factors in a chain and the active rows of the product matrix, is established. Similar correspondence between the passive rows of any two and three consecutive factor matrices in a chain is also established in Art. 49 (*cf.* what goes before in the case of a product of two matrices in Art. 42, page 163). Finally a *standard product* is defined, although it is shown in the next Article (50) that the consideration of products which are not of standard form is superfluous. The remaining articles of Chapter VI deal with the properties of passive and active rows in *any* product formed by a chain, of the conjugate of a product of any number of matrices in a specified order (*cf.* the corresponding case in Quaternions which is only a special case of Matric algebra), and of special cases. It is essentially necessary that the student studying the Calculus of matrices should fully *assimilate* the contents of this Chapter, if his further progress in that Calculus is to be assured.

Chapter VII follows Chapter VI in natural sequence. We have a chain of matrix factors. The product is a matrix bearing definite relations to the chain. But a matrix is always associated with its determinoids—that function which is the *content* of a rectangular matrix. How, then, are the determinoids of the various factors related to the determinoid of the product matrix? Dr. Cullis starts, as is the practice with him, accomplished teacher that he is, with the simplest case, *viz.*, with two factors A and B and proves the following theorems :—

(1) If the *passivity* of the product AB is *less* than its *efficiency*,

$$\det AB = 0$$

(2) If the passivity is *not less* than its efficiency η, then, $\det AB = \Sigma(-1)^{w+w'} \Delta\Delta'$ where Δ and Δ' are the determinants of two square matrices A', B', to the product of which, AB can be reduced by striking out corresponding passive rows in both the factors and active rows, in the larger factor matrix and w, w', are the affects of A', B', in A, B, respectively.

The investigation of these two theorems is divided into several possible cases including the case when the passivity is equal to the efficiency which of course falls under (2) above. The following results are deduced :—

(*a*) If A and B are both square matrices

$$\det AB = \det A \times \det B$$

(*b*) If the *smaller* of the two matrices A and B is a square matrix or if the passivity of the product is equal to its efficiency,

$$\det AB = \det A \times \det B.$$

Three *alternative* rules are given (Theorem D, page 217) for evaluating the det AB when the passivity is not less than the efficiency and it is shown how these rules can be made applicable to the other case, *viz.*, when the passivity is less than the efficiency. These rules are patterns of Art and are fully illustrated by examples. The essence seems to be the reduction of rectangular matrices to square or inversely similar (Art. 39) matrices by permissible "striking out" of rows.

Having established the above, Dr. Cullis passes on to the treatment of the more general case of the determination of the determinoid of the product of *any number* of given matrices in Art. 58. He gives six theorems, covering all possible cases but not being content with the same, gives in the next article, a practical plan for the *progressive* development of the determinoid.

Art. 60 deals with some special cases, in which one of the factors is a unit matrix, a scalar matrix or has a numerical factor. And in Art. 61 taking, without loss of generality the product of four factor matrices, A, B, C, D whose two activities are each equal to the efficiency (η) of the product, proves that

$$\det A\ B\ C\ D = \det A'\ B'\ C'\ D'$$

where A', B', C', D' are *matrices whose elements are the derived determinants of order η which can be formed from the matrices A, B, C, D respectively and where each of the extreme factors A', D', contains only one active row.*

For a proper understanding of this very important theorem which is really a concise way of expressing the rather complicated results of Theorem VI of Art. 58 (pp. 225-229), the terms "vertical and

horizontal minor matrices" are introduced in pages 233-234 and a scheme is laid down for the choice of successive minor matrices of the factors for forming the equation above.

The Theorem, the new terms and the scheme are illustrated by the working out of some examples.

The Chapter concludes with an investigation of "the algebraic sum of the affected minor determinants of order k of a product of given matrices." This is an important investigation for subsequent work and the two theorems which are given are clearly shown to be *generalizations* of those obtained in Arts. 56 and 58.

We can derive, from a fundamental matrix $[e]_m^n$ minor determinants of a given order which is less than either m or n. With these unaffected minor determinants we can construct matrices but definite schemes of formation are required. In Art. 65(1) (Chapter VII), the definition of a *complete* matrix of the minor determinants is given. According to this definition the complete matrix is $[\epsilon]_{\mu}^{\nu}$

where

$\epsilon_{pq} = (e_{mn})_k^k$ where the sequences $[m_1\ m_2\ \dots\ m_k]$ and $[n_1\ n_2\ \dots n_k]$ have their pth and qth values

$$\mu = \binom{m}{k}$$

$$= \binom{n}{k}$$

Therefore, the complete matrix can be formed in $\mu!\ \nu!\ k!\ k!$ ways; schemes of formation of complete matrices are required. This is supplied in sub-article 2. Sub-art. 3 gives what are called standard schemes of formation and standard matrices of minor determinants. In sub-article 4 complete matrices of affected minor determinants of given order are dealt with showing the *complete* character of the work under review. These definitions and formation of schemes are introductory to the two remarkable theorems which are given in Art. 66, of which, what has been given in formula (D) of Art. 61, Chapter VII, is only a particular case. The theorems are given below :-

Let $X = A\ B\ C \dots S\ T$

be a standard product in which the passivities are not subject to any restrictions. Let η be the efficiency of the product X and let K be any number which is not greater than η so that the product has derived determinants of order K.

Theorem I. If any one of the passivities is less than K, then every *complete* matrix of the minor determinants (unaffected or affected) of order K, of the product matrix X is zero.

Theorem II. If *no one* of the passivities is less than K, then every complete matrix of the minor determinants (unaffected or affected) of order K of the product matrix X, is equal to the product of any set of *correspondingly* formed complete matrices of the minor determinants (unaffected or affected) of order K of the factor matrices A, B, C ..... S, T. The nature of the correspondences alluded to in Theorem II is indicated in the proof.

Although in the enunciation the product is taken to be a standard one, it is shown that the theorems remain true when the product is not of the standard form (note 3, page 258).

Chapter VIII concludes with two theorems on the reciprocal and conjugate reciprocal of a standard product of *square* matrices. These theorems are:—

(1) The reciprocal, of a standard product, of any number of square matrices taken in a given order is equal to the product of the reciprocal of the factor matrices taken in the *same* order.

(2) The conjugate reciprocal, however, is equal to the product of the conjugate reciprocals taken in the *reverse* order. (*Cf.* Art. 54, sub-art. 2, Chapter VI.)

The proofs of these theorems depend on the general theorem of Art. 66 but an alternative proof also is given. Some instructive examples of commutative products are also given.

Rank and Connection.

Chapter IX is very important in the Calculus of matrices; and the student will do well to ponder every definition that is given and every theorem that is proved. It treats of the "Rank of a matrix and the *connections* between the *rows* of a matrix." After the student has carefully worked through the Chapter he cannot but feel grateful to Dr. Cullis for the satisfactory manner in which he has dealt with his subject.

"If $A = [a]_m^n$ be a matrix and s be any number which is not greater than the efficiency of the matrix, and, if *all* the derived determinants of order s of the matrix vanish, then all derived determinants of orders greater than s vanish; for, every such determinant can be expanded in terms of derived determinants of order, s."

"The rank of the matrix A is the greatest order which a *non-vanishing* derived determinant can have. Thus, the rank of the matrix A is γ when A has at least one derived determinant of order γ which does not vanish, whilst every derived determinant of order $\gamma+1$ vanishes and therefore every derived determinant of order greater than γ vanishes."

Obviously the rank γ of a matrix cannot exceed its efficiency, η when $\gamma < \eta$, the matrix is called "degenerate" when $\gamma=\eta$, it is called "undegenerate." An undegenerate has non-vanishing derived determinants of all orders not exceeding η but its determinoid may or may not be zero. In a degenerate of efficiency η, all derived determinants of order η vanish; also the determinoid of the degenerate vanishes; for the determinoid can be expressed in terms of the derived determinants (Art. 30, Chapter V) of order η.

"A matrix whose determinoid vanishes is called a "singular" matrix. A matrix whose determinoid does not vanish is called "non-singular." A degenerate matrix is necessarily singular and a non-singular matrix is necessarily undergenerate."

A determinoid is said to have the same rank as its matrix (Art. 68).

These preliminaries are rightly followed by twelve illustrative examples.

Art. 69 is important in as much as it establishes "*connections* between the rows of a matrix whose elements are constants," and thereby links Dr. Cullis's really original method of the treatment of matrices with the genesis of the Theory in Cayley's Memoir of 1858.

Consider the matrix $[a]_m^n$: There is said to be a *connection* between (or among) the *horizontal rows* or their matrices when there exist scalar multipliers, h_1, h_2, ... h_m, not all zero, such that

$$h_1 [a_{11}\ a_{12}\ ..\ a_{1n}] + \ ..\ h_m [a_{m1}\ a_{m2}\ ...\ a_{mn}] = \text{matrix zero.}$$

This matrix equation is equivalent to the m scalar equations:—

$$\left.\begin{aligned} h_1 a_{11} + h_2 a_{21} + \ldots h_m a_{m1} &= 0 \\ h_1 a_{12} + h_2 a_{22} + \ldots h_m a_{m2} &= 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \\ h_1 a_{1m} + h_2 a_{2m} + \ldots h_m a_{mm} &= 0 \end{aligned}\right\} (A)$$

which set can obviously be written, in Dr. Cullis's notation, as

$$[h_1 \; h_2 \; \ldots \; h_m] \; [a]_m^n = 0.$$

Similarly there is said to be a *connection* between the *vertical* rows, or their matrices, when there exist scalars $k_1 \ldots k_n$ not all zero, such that

$$k_1 \begin{vmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{vmatrix} + \ldots k_n \begin{vmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{vmatrix} =$$

matrix zero.

This is equivalent to m scalar equations among the n variables of the type

$$\Sigma k_r a_{ur} \ldots \begin{pmatrix} u = 1, 2 \ldots m \\ r = 1, 2 \ldots n \end{pmatrix}$$

which can be written in the Matrix form $[a]_1^n \begin{vmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{vmatrix} = 0.$

If n is $>$ m there will be, in the case of connection between horizontal rows $n - m$ more scalar equations of the given type but these are not dealt with for it is not the object of the learned writer to discuss, at the stage, the theory of Linear dependence; on the other hand his object is obviously to erect an *independent* Theory of Matrices on a *logical* basis and, then, to apply the same to the discussion of Linear dependence and linear algebraic equations which he does in the last Chapter of the First Volume.

Art. 70 gives eight theorems concerning *connections*. These are important for the subsequent handling of matrices. Theorem I

states that if there is any connection between the horizontal or vertical rows of a *square* matrix, the determinant of the matrix vanishes. This is fundamental and the proof is given, not by the Theory of Linear algebraic equations as is the usual procedure but by the author's Theory of matrices. Theorem II, *viz.*, that if Δ is any *non-vanishing* derived determinant of a matrix A, then the horizontal rows and also the vertical rows of A which occur in Δ are *unconnected*. This might have been a corollary to Theorem I. Theorem III is practically a generalization of Theorem II. As this theorem is important we may give its enunciation :—

"If γ is the *rank* of a matrix A and if Δ_γ is one of its non-vanishing derived determinants of order γ then ;

(*i*) the horizontal (and also the vertical rows) of A which occur in Δ_γ are *unconnected* ;

(*ii*) all other horizontal (and similarly all other vertical) rows of A are connected with those occurring in Δ_γ.

The other theorems, simple and important as showing the dependence of the *rank* of a matrix on the *connections* between rows, are easy deductions, the last Theorem, *viz.*, that "if a *connection* exists between any of the *active* rows of *either extreme* actor matrix in a product formed by a chain of matrix factors, then there is exactly the same *connection* between the corresponding rows of the product matrix," is a *generalization* of the properties of active rows considered in sub-arts. 9 and 10 of Art. 53. It is noted that the *converse* of this Theorem is not true.

Art. 71 introduces seven theorems most of which serve to facilitate the *determination of the rank of a given matrix* ; they are of fundamental importance to a student of the Calculus. In example XII, page 285, is introduced the notion of *equipotent* matrices. "Two similar matrices are called *equipotent* when either, and therefore, each, can be derived from the other by prefixing and post-fixing undegenerate square matrices. The necessary and sufficient condition is clearly set down. Equipotent matrices have equal ranks.

In note I on page 288, are given the *necessary* and *sufficient* conditions that a given matrix, A, shall have rank, γ, *viz.* :—

(1) That A shall have a non-vanishing derived determinant Δ_γ of order γ.

(2) That *every* derived determinant of order $\gamma + 1$ of A which contains Δ_γ as a minor, shall vanish.

Although these conditions follow from Theorem I of Art. 71, a more direct proof is said to be given in "Ex. V of Art. 110" and in Note 2, page 288, it is said that special theorems relating to the rank of symmetrical or self-conjugate square matrix are given in Art. 119. The First volume contains only 97 articles and these references probably relate to the second volume but the reviewer does not find that they are correct. This however has been explained on page vii of the Preface to the second volume.

Extravagant matrix.

Art. 72 deals with the rank of a product of two mutually conjugate matrices and in clause 2 of the same, is introduced the expression "*Extravagant Matrix.*" As such matrices play a considerable role, we note that "an undegenerate matrix, in which the sum of the squares of the simple minor determinants is equal to zero, is an extravagant matrix." Clearly no matrix whose elements are all real or all pure imaginaries, can be extravagant.

Art. 73 gives two theorems showing how the rank of a complete matrix of the minor determinants of a given order of a matrix can be determined, when the rank of the latter is known.

The chapter closes with some elementary investigations (Art. 74) on the rank of a matrix and connections between its rows when its elements (hitherto treated as constants) *are rational integral functions of certain variables.* The importance of this article for subsequent work cannot be over-estimated.

Chapter X deals with matrix equations of the First degree. The most general form of such an equation is

$$\Sigma_1^r A_\gamma X B_\gamma = C, \text{ where } A_1, A_2, \ldots A_\gamma, B_1, B_2, \ldots B_\gamma$$

and C are known matrices whose elements are constants. This general equation is not discussed in this Chapter but the form that is dealt with is A X B = C of which *particular* cases are A X = C and X B = C. These three receive very full treatment. In Arts. 75-77, definitions are given of a *solution*, the *general* solution, a *finite* and an *infinite* solution and equations of the specially simple form X + A = B are considered. The special cases of A X = C, X B = C and A X B = C in which A and B are *undegenerate* square matrices are first considered in Arts. 78-80 and, then, the general cases, when A and B are any matrices whatever, are discussed in the hree following articles. To a student who has studied Quaternions

the most general form $\Sigma_1^r A_\gamma X B_\gamma = C$ of the matrix Linear will recall the Quaternion equation $\Sigma A q B = C$ the operator, $\phi = \Sigma A ( ) B$ called by Sylvester, the Nivellator, and its quadrate Matrix and how Cayley has shown that the "Theory of the solution of the Linear equation in *quaternions* enters into relation with that of the solution of the Linear equation in Matrices." (*Vide* Cayley's contribution on the Analytical Theory of Quaternions in the Third Edition of Tait's Quaternions, pp. 148-152.)

Law of cancellation of matrix factors.

The chapter closes with a summary of cases in which it is allowable to *cancel* a matrix factor which is *common* to both sides of a matrix equation. Art. 84 gives very *precise statements* (six in number) which should be carefully followed and borne in mind, regarding the *Law of Cancellation* of matrix factors in a matrix equation. We notice in the enunciation of the Law the importance of the notions of *rank* and *passivity*. Art. 85 follows, in natural course, with similar statements regarding the cancellations of matrix factors in a matrix *Identity* which too receives a definition. A thorough comprehension of the Law of Cancellation is essential to progress in the General Theory of rectangular matrices.

Solutions of any system of Linear algebraic equations by the help of the Theory of matrices.

Chapter XI, the last in the First Volume, treats of the solution of *any system* of Linear algebraic equation. This is a really valuable Chapter and the reviewer has not seen this important subject dealt with in the same masterly and refreshing manner in any of the modern English works in Algebra. The late Maxime Bocher's Introduction to Higher Algebra, the best of its kind in recent years, gives only the *beginnings* of the general theory. Starting with *connections* (clearly defined) between linear *functions* and between linear *equations* (Art. 86), Dr. Cullis gives, as a first treatment, the solution of any system of linear equations and writes the *single* matrix equation which is equivalent to the n scalar equations of the type:—

$$a_{11} x_1 + a_{12} x_2 + \dots a_{1n} x_n = c_1.$$

Since there is one—one correspondence between the solutions of the system of scalar equations and the solutions of the matrix equation, the solutions of the scalar equations depend on the solutions of a matrix equation of the first degree of the form $A X = C$,

discussed in the previous Chapter. Stating clearly what are the *unaugmented* and *augmented* matrices of the system of the scalar equations, the theorems arrived at, in regard to the matrix equation, in Art. 81 of Chapter X, are translated to apply to the scalar equations and six noteworthy results are given including the *infinite* solutions. The author next proceeds to find the *general* solution when finite solutions exist and discusses two cases, the principles being fully illustrated. A second treatment of the solution of a system of Linear equations of the type :—

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + a_{1n+1} = 0$$

is given in Art. 88 and similarly dealt with. The next article discusses the solution of any system of *homogeneous* linear equations which are of such frequent occurrence in analysis. In Arts. 90-93 *complete sets* of unconnected solutions of any system of linear algebraic equations are defined and by means of them a symmetrical form is given to the general solution. The various examples worked out in connection with Art. 90, introduce the ideas of 'extravagant' and "non-extravagant" equations and solutions, although the reviewer would have wished an "Extravagant solution" to be *formally* defined as an "Extravagant equation" is in Ex. vii, page 389. In Arts. 94 and 95 properties of mutually *orthogonal* solutions are considered and it is shown, in particular, that a system of linear equations, whose matrix is *real*[1] (*i.e.*, a matrix all of whose elements are real) has always a *complete set* of unconnected mutually orthogonal real solutions. Connections between linear equations receive *still another interpretation* in Art. 96 and the last Chapter, which is probably the only chapter in the book which contains *applications* of the Theory discussed in the previous Chapters, closes with a discussion of the conditions for the existence of *functional* dependences between algebraic functions.

[1] The idea of a domain of rationality Ω should be fully grasped, in order to understand what a *general* theorem concerning matrices becomes in the special case when the matrices are *real*. When the operations on the elements are only *rational* the general theorem can be made at once applicable to the real matrix by taking Ω to be the domain of all real numbers. But it should be noted that the real domain has properties which are not possessed by other domains and special treatment is necessary. (This subject has been further developed by Dr. Cullis in Chapters XVI and XVII of his second volume.)

The Second Volume.

The Second Volume has been published in 1918. "The continuation of the work" we are told in the interesting preface "had been greatly hindered by untoward circumstances, above all by the difficulty of obtaining sufficient leisure for the final preparation of the manuscript for the Press." The theoretical portion has been constantly increased, in the first place by *abstractions* from the applications and, in the second place by incorporating the work of other writers. The learned author decided, with reluctance (for, his original aim was "to give a systematic account of certain *applications* of Matrices to illustrate the very great advantages gained by using them in almost all branches of Mathematics"), to publish, as a Second Volume, the first half of that portion of the complete work which deals in greater detail with the Theory of Matrices. The Second Volume, accordingly, contains those parts of the Theory which naturally precede any investigation of the special properties of functional Matrices (Matrices whose elements are rational integral functions of a finite number of variables). It deals almost exclusively with Matrices whose elements are constants, which may be arbitrary parameters, and with those transformation of such Matrices which are classed as equigradent.

The Volume contains eight Chapters, and like the first, has a full and useful index, and a summary of the contents of each Chapter prefixed to it. There are three appendices,—Appendix A, on the utility of the relations obtained in Chapter XIII, *viz.*, between the elements and minor determinants of a matrix; Appendix B, on the Pfaffian of a Skew-Symmetric matrix of even order; which was suggested to Dr. Cullis by his perusal of a paper by Mr. Haripada Datta (on Symmetric determinants and Pfaffians, Research Paper No. 5 in the Mathematical Department of the University of Edinburgh) and Appendix C,—on equigradent transformations in which one of the transforming factors is a semi-unit matrix.

Dr. Cullis has usefully anticipated the reviewer, by giving, on pp. viii-x, remarks concerning the contents of the *individual* Chapters. In the Preface to the Second Volume, he has taken the student of Higher algebra more into his confidence and admitted him into his work-shop so that he may take a glance at the materials with which he is raising his edifice—not finished yet but of magnificent

proportions and bidding fair to end in an outstanding land-mark in the Mathematical literature of the present century!

We note that although "the present volume is avowedly restricted to the *Theory* of matrices, it actually presents a large number of Geometrical *Applications* and of algebraical *Applications* to which, it is promised, attention will be subsequently directed." Thus, "any set of matrices which are *vertically equivalent* to one another are regarded as defining a "Spacelet,"[1] which is completely represented by *any one* of them, usually by one which is undegenerate, the "Spacelet" being a "point" when the common rank of the matrices is 1. And any property of a matrix which remains unaltered when the matrix is replaced by any matrix vertically equivalent to it, is a property of a "Spacelet" and conversely all properties of "Spacelets" are properties of "matrices." "Spacelets" and "points" are spoken of, from the outset, as *geometrical* concepts although their *definitions* are not so. The object expressly is to "lighten subsequent Chapters on the geometrical applications of matrices" such as to n-dimensional geometry. In Chapter XVI which is on '*equigradent* transformations,' it is shown that "every such transformation of a matrix whose elements are constants, corresponds to a *linear* transformation of the variables in a *bilinear* or *quadratic* algebraic form (Note 6, page 234). This is a note-worthy application of the Calculus of matrices, on the algebraic side.

On the geometrical side, the notion of "Spacelets" plays a great part in the Second Volume. The 'rank' of a 'spacelet' the 'paratomy' of two spacelets and the relations between 'spacelets' represented by the terms '*intersection*' and '*join*' and by the terms '*incident*' and '*connected*,' are properties which depend only on the notion of *connection* and are *invariant* in every equigradent (or projective) transformations of the points of space. The "extravagance" of a 'spacelet,' the 'orthotomy' and 'cross-rank' of two 'spacelets' and the relations between 'spacelets' represented by the terms 'core' and 'plenum,' 'orthogonal' and 'normal,' are properties which depend on the notion of '*orthogonality*' and are invariant in every

[1] In generalized Vector analysis, a matrix of n short rows and m long rows may be considered as given by m Vectors of an n-dimensional space and there is no doubt many of the theorems arrived at may be obtained expeditiously from the point of view of such analysis but the author's stand-point is quite *different* and this fact must not be over-looked by a reviewer.

'*semi-unit transformation*' of the points of space, *i.e.*, equigradent transformation which leaves the Absolute Quadric unaltered. These are indeed valuable contributions to the much-looked for *applications* of the Calculus of matrices and as, aids to the imagination, which is often baffled in dealing with hyper-space problems, 'spacelets' and their intersections are graphically represented by areas, a *shaded* area always representing a *completely extravagant* spacelet, *i.e.*, a spacelet which being *orthogonal* with itself, is a generator of the Absolute Quadric.

Compound and Compartite matrices.

The opening Chapter (XII) which deals with 'compound' and 'compartite' matrices, gives notations for such matrices and their determinoids which are the complete generalizations of the notations used in the First Volume.

Notation.

A convenient and expressive notation is of the first importance, alike to the mathematician who propounds a Theory and to the student, who tries to understand him. Liebniz recognised early the value of form. Notations and algorithms held a prominent place in his conceptions and he wrote in 1676 that "one part of the secret of analysis consists in the *characteristic*, *i.e.*, to say in the Art of employing well the symbols we use. As Laplace said "the *language* of analysis, most perfect of all, being a powerful instrument of discoveries, its notations, specially when they are necessary and happily conceived, are so many germs of new calculi." In the development of a subject like the matrix theory, sufficient stress cannot be laid on the importance of a well-chosen notation. "It rests with the worker in mathematics not only to explore new truths but to devise the language by which they may be discovered and expressed; and the genius of a great mathematician displays itself no less in the notation he invents for deciphering his subject than in the results attained." And one cannot but have faith in "the power of well-chosen notation to simplify complicated theories and to bring remote ones near." The notations for matrices and determinoids which Dr. Cullis has elaborated in his work, fulfil the requirements of a useful instrument of research and are certainly an improvement on the notations of Cayley and of other previous writers.

Relations between the elements and the minors.

Full of interesting matter is Chapter XIII, dealing with the relations between the elements and minor determinants of a matrix. Starting with the determination of the connections between the short

rows of an undegenerate matrix (Art. 108), we are led to a host of useful relations which are finally seen to be deducible from Art. 116 headed "Identical relations between the elements of any matrix." In Arts. 110 and 111 are given *Sylvester's identities* (a reference to the Collected works of Sylvester where these occur had been useful) involving the "primary superdeterminants" (*vide* note 1 on page 17, Vol. II) of a determinant and the corresponding identities involving the "primary sub-determinants" (also defined in Note I on page 17) of a determinant. Conditions for the *equivalence* of two similar undegenerate matrices and the *equivalence* of two systems of linear algebraic equations are given in Arts. 113 and 114, the sign '$\equiv$' being introduced to denote the equivalence of two matrices. Note 3 to Art. 113, pages 78-80 is specially interesting as introducing the notion of undegenerate matrices as "spacelets." "An undegenerate matrix $\underset{\alpha\times n}{a}^{\gamma}$ of rank γ where γ is $< n$ represents a flat locus of $\gamma-1$ dimensions in homogeneous space of $n-1$ dimensions, *i.e.*, it represents a certain $(\gamma-1)$-way subspace or 'spacelet' w_γ of the complete $(n-1)$-way homogeneous space w_n. γ is called the rank of the 'spacelet' w_γ and, in particular n is the rank of the complete space w_n. The elements of the successive γ vertical rows of $\underset{\alpha\times n}{a}^{\gamma}$ are the projective coordinates of γ unconnected points of w_n which lie in w_γ and completely determine w_γ. Every other similar undegenerate matrix $\underset{\alpha\times n}{b}^{\gamma}$ which is equivalent to $\underset{\alpha\times n}{a}^{\gamma}$, *i.e.*, when $\underset{\alpha\times n}{b}^{\gamma} = \underset{\alpha\times n}{a}^{\gamma}\,\underset{n\times\gamma}{k}^{\gamma}$ where $\underset{n\times\gamma}{k}^{\gamma}$ is undegenerate, represents the same spacelet w_γ. Examples IV and V of Art. 113 are of 'spacelets' in homogeneous 3 and 4 dimensional spaces.

Spacelets.

Chapter XIV gives an account of some special properties of a product of *square* matrices. The Chapter opens (Art. 118) with the recapitulation of the properties of a product of square matrices already proved in Vol. I. Art. 119 defines two *cojoint complete* matrices of the minor determinants of a square matrix. In this and some of the following articles are given a number of their properties. Art. 124 gives a separate account of the properties of the reciprocal of a square matrix and the next three articles deal with properties of symmetric and Skew-symmetric matrices (Appendix B should also be read

Square matrices.

in connection with these) criteria for the determination of their ranks being given. Anent Skew-symmetric matrices the reviewer would like to invite attention to the following prize problem announced in 1919 by the Philosophical faculty of the University of Berlin :—

Skew-Symmetric matrices.

"To determine by means of the *Theory of Elementary divisors*, the criteria that a given matrix be capable of representation as the composition (product) of two Skew-symmetric matrices."

A Prize-problem.

(*N.B.*—Prof. A. A. Bennett discussed the problem in the Bulletin of the American Mathematical Society for July 1919, pp. 455-458. He has acknowledged, in March 1920 number of the Bulletin, the reviewer is glad to notice, Harold Hilton's prior solution in Section 8, Chapter VI of his well-known work on "Homogeneous Linear substitutions").

Chapter XV deals with the possible *ranks* of the product matrix and the factor matrices, in any matrix product. "The theorem of Art. 133 and the final results of Articles 135 and 137 constitute the *complete generalization of the Law of cancellation for matrices.*" The concluding articles deal with the *equivalences* of matrices on which the definition of a 'spacelet' is based and with the "joins," "intersections" and "connections" of matrices and spacelets. These require a careful study as introductory to the geometrical applications of the Calculus of matrices.

Law of Cancellation of matrices—its generalization.

Chapter XVI is on equigradent transformations of matrices and the use thereof in the reduction of a matrix of rank γ to standard forms. The transformation of $[a]^{\alpha}_{\rho}$ into $[b]^{\alpha}_{\mu}$ by means of an equation of the form $[b]^{\alpha}_{\mu} = [h]^{\rho}_{\mu}\ [a]^{\alpha}_{\rho}\ [k]^{\alpha}_{\alpha}$ in which $[h]^{\rho}_{\mu}$ and $[k]^{\alpha}_{\alpha}$ are undegenerate matrices with constant elements having ranks r and s equal to their respective passivities, is called an *equigradent* transformation. This definition not only holds when the matrices have constants as their elements but also when they have *rational integral functions of certain variables as their* elements. In the latter case it can be shown that an equigradent

Equigradent transformations.

transformation leaves the minimum degrees of connection of the matrix transformed unaltered. Hence the name 'equigradent.' Two equigradent matrices have necessarily the same rank.

The simplest standard form is a similar matrix which is conventionally equal to the unity matrix $[1]_r^r$ and it is shown that every matrix of rank r with contant elements, can be obtained from the unit matrix, by an equigradent transformation, which reminds one of the development of the Theory of integers from unity. The reductions of symmetric and Skew-symmetric matrices by symmetric equigradent transformation receive careful attention. This is an important and difficult Chapter and the student will do well to grasp thoroughly the part played by the notion of Domain of rationality Ω, in the transformations.

Matrix equations of the Second degree.

Chapter XVII is on the solution of matrix equations of the second degree. "One of the most important results obtained in this Chapter, is the *general* formula for all solutions of any assigned rank ρ of the symmetric equation $\underset{\sqcup m}{x}\,[x]_s^m = \underset{\sqcup m}{a}\,[a]_m^r$ where $[a]_r^m$ is a given matrix of rank r (Art. 157). This is important, in as much as, the general Theory of *extravagant* matrices is chiefly based on this result and leads at once to the reductions in Chapter XVIII.

The reviewer cannot pass without notice the important notions introduced in Art. 155 of a "Semi-unit matrix" and of a "square semi-unit matrix." The examples attached to this Article are specially interesting as showing the power and elegance of the Calculus when applied to such familiar things as rotations of a rigid body in 3-way space.

Extravagance and degeneracy.

Chapter XVIII, the last but one Chapter in the Second Volume, deals with the very important notions of "the degeneracy" and "the extravagance" of a matrix. The "degeneracy" is the amount by which the rank of a matrix falls short of its "efficiency." In sub-article 2 of Art. 72 in the First Volume, an "extravagant matrix" had been defined but it is necessary to define what is the "extravagance" *of* a matrix. It is the degeneracy of the product $\psi = [a]^n$

$\underset{n}{\overset{r}{\lceil\!\lfloor a \rfloor\!\rfloor}}$. The "extravagance" of a spacelet (or the degree of its orthogonality with itself) defined in Art. 170 is that property of it which is next in importance to its "rank." It is invariant in every semi-unit transformation of the points of space and can be interpreted as being the "rank" of contact of the 'spacelet' with the Absolute Quadric. The Chapter deals with interesting geometrical notions, the "core" and "plenum" of a 'spacelet' being defined. With every 'spacelet' is associated a completely *extravagant* "spacelet" called its 'core' which is the locus of all points which lie in the given spacelet and are orthogonal with it, *i.e.*, the locus of the points in which the given spacelet touches the Absolute Quadric and a *plenarily extravagant* (vide note 2, page 382) spacelet called its "*plenum*" which is the smallest spacelet containing the given spacelet and all points orthogonal with it. Mutually "orthogonal" spacelets are distinguished from mutually "normal" spacelets. A given spacelet has *one* and only *one* normal while an *indefinite number* of spacelets are orthogonal with it.

On the algebraic side the reduction of a matrix whose elements are constants to a standard form by a unilaterally semi-unit equigradent transformation is specially valuable, as it enables us to complete the discussion of the unconnected mutually orthogonal solutions of any system of homogeneous linear algebraic equations which had been left unfinished in the closing Chapter of the First Volume.

Chapter XIX is on the "Paratomy" and "orthotomy" of two matrices and of two spacelets of homogeneous space. Two matrices $\underset{n}{\overset{p}{\lceil\!\lfloor a \rfloor\!\rfloor}}$ and $\underset{n}{\overset{q}{\lceil\!\lfloor b \rfloor\!\rfloor}}$ of ranks r and s which have the same number of vertical rows or two matrices $[a]^{n}_{p}$ and $[b]^{n}_{q}$ of ranks r and s which both have the same number of horizontal rows, are to be regarded as having the same mutual 'paratomy' and the same mutual 'orthotomy' as the homogeneous spacelets $w_r \equiv \underset{n}{\overset{p}{\lceil\!\lfloor a \rfloor\!\rfloor}}$ and $w_s \equiv \underset{n}{\overset{q}{\lceil\!\lfloor b \rfloor\!\rfloor}}$. The author, therefore, confines himself to a consideration of the paratomy and orthotomy of spacelets in homogeneous space. The mutual paratomy, p_1 of any two spacelets w_r and w_s in space w_n is defined to be the greatest number of unconnected points common to w_r and w_s. It is the rank of that

Paratomy and orthotomy.

spacelet which is the locus of all points common to w_r and w_s, *i.e.*, the rank of the complete intersection of w_r and w_s. The *mutual orthotomy* of any two spacelets $w_r \equiv \overbrace{a}^{r}$ and $w_s \equiv \overbrace{b}^{s}$ of space w_n is defined to be the *degeneracy* of the product $\psi = [a]_r \underbrace{b}_{s}$ and the rank of ψ is called the *cross rank* of spacelet w_r and w_s. The most moteworthy results are those relating to the *greatest* possible mutual 'orthotomy' of two spacelets and the *independence* of the extravagances of two spacelets of given ranks which have a given complete intersection. "Real spacelets" are dealt with in Articles 181 and 182 (notes appended thereto) and it is easy to see that all the abstract theorems of the last Chapter are applicable to the common metrical space Ω_{n+1} of n dimensions when we define the paratomy and orthotomy of two spacelets of this space, to be those of their infinite sub-spaces, *i.e* those of their intersections with the infinite sub-space w_n of the space of n dimensions. The Chapter closes (Art. 183) with a treatment of the properties of mutually orthogonal spacelets.

Applications of the Theory of Matrices.

As Sylvester said "the theory of determinants is an algebra upon algebra, a Calculus which enables us to construe and foretell the results of algebraical operations in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic." "Properly regarded, the theory of determinants" of which so much could be said, is as remarked by M. Bocher, "hardly more than a shadow of the theory of matrices and just so far as one attempts to ignore this fact does the subject become an artificial one or at best a tool for other investigations." Even as a tool, its potency is great; for, we find the theory largely entering into modern works on Algebra, Geometry and Applied Mathematics. The reviewer may be pardoned if he would mention below some few of the applications of the theory which the learned author has developed with such consummate skill :—

(*i*) The reviewer has come across a notice of a remarkable book on Analytical Geometry entitled "Cinq Etudes de Geometrie analytique" by M. Stuyvaert. Gand. E. Von Goethem, 1908, vi & 230 pp. This work consisting of applications of the Theory of matrices and elimination was awarded the "Prix François Deruyts" by the

Royal Academy of Belgium. It exhibits the *singular power* and *elegance* of the Matrix theory and has received an interesting review, in pp. 479-484 of the Bulletin of the American Mathematical Society, Vol. 17 (1910-11).

(*ii*) The late M. Bocher's Introduction to Higher Algebra gives some applications of the Theory. The last three chapters of the the book, constitute an excellent study of matrices whose elements are polynomials in a single variable λ called λ matrices. The theory is applied (*a*) to the elucidation of the Theory of Elementary divisors invented by Sylvester, H. J. S. Smith and more particularly by Weirstrass and perfected by Kronecker, Frobenius and others, (*b*) to the equivalence and classification of bilinear forms and collineations and (*c*) to the equivalence and classification of pairs of Quadratic forms.

(*iii*) Prof. H. J. S. Smith's Memoir on the "arithmetical invariants of a rectangular matrix" (*vide* his collected works) is one of the earliest known applications to the Theory of numbers.

(*iv*) Application of the Theory to Linear Differential equations by J. Brill (*vide* Cambridge Philosophical Society's Proceedings, Vol. 8). This memoir is written on the basis of Cayley's developments.

(*v*) O. Veblen and Young's Projective Geometry makes much use of the Matrix Theory (*vide* Art. 54, page 156, Art. 95, pages 268 and 269 of Vol. I, and Arts. 124 *et seq* of Vol. II). There are, in this work, exceedingly interesting applications of the elements of the Theory of matrices.

(*vi*) Miller and Blichfieldt's "Finite groups," Art. 6, pp. 13-14 discusses the bearing of the theory of matrices on the Theory of groups, in an elementary way.

Blichfieldt's Finite Collineation groups, Chapter I gives applications of the theory *to Liner groups.*

(*vii*) The elegant article by Dr. Cullis on "a special square matrix of order six" in the Bulletin of the Calcutta Mathematical Society, Vol. X, No. 3, gives by means of the theory, the stress-strain relations of an isotropic body without assuming the existence of a strain-energy function.

The reviewer feels that he has only imperfectly sketched the contents of the first two volumes of a remarkable contribution to the ever-growing mathematical literature of the twentieth century—remarkable

alike for its breadth and depth, for its logical structure and its didactic value.[1] The labour involved by and the earnestness of purpose brought to bear on, the making of the work have been worthy of the author whose reputation as an accomplished Mathematician and as an inspiring teacher has reflected glory on the University of Calcutta, where he holds now the exalted position of the Hardinge Professor of Mathematics. This appointment, as he tells us in the Preface to to the Second Volume, had afforded him additional leisure for the completion of the monumental work he has undertaken and it is a pleasure to note the author's expression of special gratitude to the Hon'ble Sir Asutosh Mookerjee, Kt., C.S.I., President of the Post-Graduate Councils in Arts and Science for his stimulating interest and encouragement—interest and encouragement in a really promising field of work.

Elaborate character of the work.

But the very character of the work in its elaborateness and its novel nomenclature renders it necessary that a manual for students attacking the subject for the first time should be written. Such a manual may well be on the lines of Joly's Manual of Quaternions, in the first 20 pages of which is given the 'grammar' of the Calculus. Gauss writing to Bessel in 1810 lamented "I am giving this winter two courses of lectures to *three* students, of which one is only moderately prepared, the other less than moderately and the third lacks both preparation and ability. Such are the onera of a mathematical profession." A handy manual interspersed with obvious applications is likely to remove the possibility of such a lament.

[1] The Reviewer feels that the Theory of Matrices when completely developed is likely to include all other mathematical disciplines.

Note on Spherical Waves of Finite Amplitude

By

Sudhansukumar Banerji

The equations governing the mode of propagation of aereal plane waves of finite amplitude were first obtained by Poisson [1] as long ago as 1808, and formed the subject of further investigation by Stokes [2] in 1848. In the case of the adiabatic propagation the exact equation was first derived by Earnshaw [3] in the form

$$\left(\frac{\partial y}{\partial x}\right)^{\gamma+1}\left(\frac{\partial^2 y}{\partial t^2}\right)=c\quad\frac{\partial^2 y}{\partial x^2}$$

The mode of propagation of waves according to this equation has been discussed by Riemann, [4] Rankine [5] and Hugoniot. [6] In an important memoir, Rayleigh [7] gave a critical and historical account of the subject and discussed in further detail special solutions of the problems of finite waves and the influence of viscosity and thermal conductivity on their mode of propagation. It will thus appear that although the aerial plane waves of finite amplitude have received a fairly adequate treatment, the propagation of spherical waves of finite amplitude has not hitherto received any attention inspite of the fact that the differential equations for both types of disturbances bear a close resemblance to each other. In the following pages the principal features of the latter type of disturbance have been obtained.

[1] Poisson "Mémoire sur la Théorie du Son." *Journ. de l' Ecole Polytechnique*, Vol. VII, p. 319, *et. seq.*, 1808.

[2] Stokes, "On a difficulty in the theory of sound," *Phil. Mag.*, 1848. *Mathematical and Physical Papers*, Vol. II, p. 51.

[3] Earnshaw, *Roy. Soc. Proc.* 1859; *Phil. Trans. Roy. Soc.*, 1860, p. 133.

[4] Riemann, Werke, p. 157. *Gott. Abh.*, t VIII. p. 43 (1858-59).

[5] Rankine, *Mis. Sc. Papers*, p. 530, *Phil. Trans.*, Vol. 160, p. 277, 1870.

[6] Hugoniot, *Journ. de l' Ecole Polytechnique* (1887).

[7] Rayleigh, 'Aerial Plane waves of Finite Amplitude,' *Proc. Roy. Soc.*, A, Vol. 84 p p. 247-284 (1910).

As in the case of plane waves the same difficulty will of course be experienced on account of the motion becoming ultimately discontinuous. For a detail discussion of the question of discontinuity for plane waves reference may be made to Taylor [1] and Wilton. [2]

That the ordinary approximate theory of infinitesimal aerial waves is incapable of explaining many of the phenomena observed with waves of great intensity such as that which accompanies a sudden explosion, a gun-fire or an electric discharge has been long recognised by the physicists. Quite recently Dr. Louis Vesot King [3] has given an interesting application of the theory of plane waves of finite amplitude in a discussion of the acoustic efficiency of fog-signal machinery. In a subsequent paper, as an application of the theory of spherical waves of finite amplitude, a discussion will be given of the explosion waves arising from the sudden explosion of a detonating gas contained within a spherical envelope.

The Eulerian equations for spherical disturbances are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}, \qquad \ldots (1)$$

$$\frac{\partial \rho'}{\partial t}+u\frac{\partial \rho'}{\partial r}=-\frac{\partial u}{\partial r},$$

where $\rho'=\log \rho r^2$, ρ denoting the density.

The equations when written in this form are remarkably similar to the equations for plane waves of finite amplitude, *viz.*,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial \rho_1}{\partial t}+u\frac{\partial \rho_1}{\partial x}=-\frac{\partial u}{\partial x},$$

where $\rho_1=\log \rho$, ρ being the density.

To determine the laws of propagation, we use Riemann's method. If we put

$$P=f(\rho')+u, \quad Q=f(\rho')-u,$$

[1] Taylor, *Proc. Roy. Soc.*, 1910.

[2] Wilton, *Phil. Mag.*, 1913

[3] Louis Vesot King, "On the Propagation of Sound in the free atmosphere and the acoustic efficiency of fog-signal machinery, *Phil. Trans.*, Vol. 218, 1919.

where $f(\rho')$ is as yet undetermined, we obtain by differentiation

$$\frac{\partial P}{\partial t} = f'(\rho') \frac{\partial \rho'}{\partial t} + \frac{\partial u}{\partial t},$$

$$\frac{\partial P}{\partial r} = f'(\rho') \frac{\partial \rho'}{\partial r} + \frac{\partial u}{\partial r},$$

$$\frac{\partial Q}{\partial t} = f'(\rho') \frac{\partial \rho'}{\partial t} - \frac{\partial u}{\partial t},$$

$$\frac{\partial Q}{\partial r} = f'(\rho') \frac{\partial \rho'}{\partial r} - \frac{\partial u}{\partial r}.$$

Therefore

$$\frac{\partial P}{\partial t} + u \frac{\partial P}{\partial r} = \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) + f'(\rho') \left(\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial r} \right)$$

$$= -\frac{1}{\rho} \frac{\partial p}{\partial r} - f'(\rho') \frac{\partial u}{\partial r}.$$

If now we determine $f(\rho')$ so that

$$\{f'(\rho')\}^2 \frac{\partial \rho'}{\partial r} = \frac{1}{\rho} \frac{\partial p}{\partial r},$$

we get

$$\frac{\partial P}{\partial t} + u \frac{\partial P}{\partial r} = -\left[f'(\rho') \frac{\partial \rho'}{\partial r} + \frac{\partial u}{\partial r} \right] f'(\rho')$$

$$= -f'(\rho') \frac{\partial P}{\partial r}.$$

Similarly,

$$\frac{\partial Q}{\partial t} + u \frac{\partial Q}{\partial r} = f'(\rho') \left(\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial r} \right) - \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right)$$

$$= -f'(\rho') \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$= f'(\rho') \left[f'(\rho') \frac{\partial \rho'}{\partial r} - \frac{\partial u}{\partial r} \right]$$

$$= f'(\rho') \frac{\partial Q}{\partial r}.$$

Thus we obtain

$$dP=[dr-\{f'(\rho')+u\}dt]\frac{\partial P}{\partial r},$$

$$dQ=[dr+\{f'(\rho')-u\}dt]\frac{\partial Q}{\partial r}.$$

Hence in spherical waves of finite amplitude $dP=0$, or P is constant, for a geometrical point moving with the velocity

$$\frac{dr}{dt}=f'(\rho')+u$$

$$=\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+u, \qquad \ldots (2)$$

whilst Q is constant for a point moving with the velocity

$$\frac{dr}{dt}=-f'(\rho')+u$$

$$=-\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+u. \qquad \ldots (3)$$

Hence any given value of P moves forward, and any value of Q moves backward, with the velocity given by (2) or (3) as the case may be.

Now since $\left[\frac{1}{\rho}\frac{dp}{d\log(\rho r^2)}\right]$ can also be written in the form

$$\left[\frac{1}{\rho}\frac{dp}{d\rho}\frac{1}{\frac{1}{\rho}+\frac{2}{r}\frac{dr}{d\rho}}\right],$$

the above expressions for the velocities of propagation for the forward and the backward motion will reduce when r is very large to the well-known expressions

$$\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}+u \text{ and } -\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}+u$$

for these quantities for plane waves of finite amplitude just as we should expect.

As in the case of Riemann's method for plane waves of finite amplitude, the above results well enable us to understand in a general way,

the nature of the motion in any given case. Thus if the initial disturbance be supposed to be confined between the two spherical surfaces $r=a$, $r=b$, we may suppose that P and Q both vanish for $r>a$ and for $r<b$. The region within which P is variable will advance, and that within which Q is variable will recede, until after a time these regions separate and leave between them a space for which P=0, Q=0 and in which the fluid is therefore at rest. The original disturbance has thus been split up into a diverging and a converging wave travelling in opposite directions. In the advancing wave we have Q=0 and therefore $u=f(\log \rho r^{\kappa})$. It ill thus appear that in spherical waves of finite amplitude both the density and the particle velocity are not propagated forwards at the same rate as in the case of the plane waves of finite amplitude. The velocity of propagation will however be found to be greater, the greater the value of ρ. If we draw a curve with r as abscissa and ρ as ordinate, and make each point of this curve move forward with the appropriate velocity, those parts of the curve will move faster which have the greater ordinates and the curve will eventually become at some points perpendicular to r making the quantities $\frac{du}{d\rho}$, $\frac{d\rho}{dr}$ infinite and giving rise to the well known case of the wave of discontinuity.

It is interesting to trace the variation of the velocity of propagation at different distances from the source. For this purpose it is convenient to write the quantity $\frac{1}{\rho}\frac{d\rho}{d\log \rho r^{\kappa}}$ in the form $\frac{d\rho}{d\rho}\cdot\frac{1}{1+2A\rho/r}$ where $A=\frac{dr}{d\rho}$.

Whether at any point the velocity of propagation so far as it is dependant on this quantity is less or greater than the *standard* velocity $\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}$ obviously dependson the sign of A, that is, on the sign of $\frac{d\rho}{dr}$. If the wave be supposed to start from a spherical surface of small radius a as a wave of condensation followed by rarefaction, then initially $\frac{d\rho}{dr}$ is negative and the initial velocity is greater than the standard velocity. If on the other hand it starts as a wave of rarefaction followed by condensation, the initial velocity is less than the standard velocity. If discontinuity does not set in at any stage of the motion, then the velocity of propagation approaches the standard velocity at a considerable distance from the origin.

Using the Lagrangian method if ξ denotes the displacement at time t of the particle whose undisturbed abscissa is r and if we write

$y=r+\xi$, so that y denotes the absolute coordinate at time t, the dynamical equation obtained by considering the forces acting on a unit area of a stratum formed by two consecutive spherical surfaces is

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = -\frac{\partial p}{\partial r} ,$$

where ρ_0 is the density in the undisturbed state and the equation of continuity is

$$\rho_0 r^2 = \rho y^2 \frac{\partial y}{\partial r} ,$$

ρ being the density in the disturbed state.

If as before we write $\rho' = \log \rho y^3$, and if we suppose that the pressure is a function of ρ only, the dynamical equation can be written in the form

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = -\frac{dp}{d\rho'} \cdot \frac{\partial \rho'}{\partial r} .$$

Also since $\rho' = \log \rho y^3$, we have $\frac{\partial \rho'}{\partial r} = \frac{1}{\rho y^3} \frac{\partial (\rho y^3)}{\partial r}$. Also from the equation of continuity $\rho_0 = 3\rho y^2 \frac{\partial y}{\partial r}$ and therefore $-\frac{\rho_0}{3(\rho y^3)^2} \frac{\partial (\rho y^3)}{\partial r}$ $= \frac{\partial^2 y}{\partial r \partial r^3}$.

Consequently $$-\frac{\rho_0}{3\rho y^2} \frac{\partial \rho'}{\partial r} = \frac{\partial^2 y}{\partial r \partial r^3} .$$

The dynamical equation can therefore be written in the form

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \frac{dp}{d\rho'} \cdot \frac{3\rho y^2}{\rho_0} \frac{\partial^2 y}{\partial r \partial r^3} = \frac{dp}{d\rho'} \cdot \frac{\dfrac{\partial^2 y}{\partial r \partial r^3}}{\dfrac{\partial y}{\partial r^3}} .$$

That is to say,

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho} \frac{dp}{d(\log \rho y^3)} \cdot \frac{\dfrac{\partial^2 y}{\partial r^3 \partial r^3}}{\dfrac{y^2}{r^2}\left(\dfrac{\partial y}{\partial r^3}\right)^2} .$$

It is easy to express the differential equation also in the form

$$\frac{\partial^2 y}{\partial t^2} = \frac{dp}{d\rho} \cdot \frac{\dfrac{\partial}{\partial r}\left(\dfrac{y^2}{r^2}\dfrac{\partial y}{\partial r}\right)}{\dfrac{y^4}{r^4}\left(\dfrac{\partial y}{\partial r}\right)^2} \qquad \ldots \quad (4)$$

that is to say in the form

$$\frac{\partial^2 y}{\partial t^2}=-\frac{dp}{d\rho}\frac{\partial}{\partial r}\left[\frac{1}{\frac{y^2}{r^2}\frac{\partial y}{\partial r}}\right].$$

A general solution of this differential equation is not easy to obtain An asymptotic solution can however be easily obtained. At a great distance from the origin, that is say, when r is large, this equation reduces to the form

$$\frac{\partial^2 y}{\partial t^2}=\frac{dp}{d\rho}\cdot\frac{\frac{\partial^2 y}{\partial r^2}}{\left(\frac{\partial y}{\partial r}\right)^2},$$

which is of the same form as the Earnshaw's differential equation. The general integral of this latter equation is obtained by eliminating a between the equations

$$y=ar+((C\pm c\log a)t+\phi(a)$$

$$0=ar\pm ct+a\phi'(a),$$

where C is a constant, ϕ is arbitrary and $c^2=\frac{dp}{d\rho}$. This is as we should expect for the spherical disturbances ultimately vanish as a set of plane waves.

Earnshaw's method can be adopted to obtain a general integral of this equation if it is possible to obtain a function τ, such that

$$\frac{\partial y^3}{\partial \tau}=\frac{1}{\sqrt{3}r}\frac{\partial y}{\partial t},$$

τ being independent of y.

For writing the differential equation in the form

$$\frac{1}{3r^2}\frac{\partial^2 y}{\partial t^2}=\frac{dp}{d\rho}\cdot\frac{\frac{\partial^2 y^3}{\partial (r^3)^2}}{\left(\frac{\partial y^3}{\partial r^3}\right)^2}$$

and making the assumption

$$\frac{\partial y^3}{\partial \tau}=F\left(\frac{\partial y^3}{\partial r^3}\right).$$

we easily obtain the first integral

$$\frac{\partial y^3}{\partial \tau} = C \pm c \log \frac{\partial y^3}{\partial r^3} = \frac{u}{\sqrt{3r}},$$

and the general integral is obtained by eliminating a between the two equations

$$y^3 = ar^3 + \frac{u}{\sqrt{3r}}\tau + \phi(a)$$

$$0 = ar^3 \pm c\tau + a\phi'(a),$$

where as before $c^2 = \frac{dp}{d\rho}$ and ϕ is arbitrary.

On the Theory of continued Fraction[1]

[Third Paper]

By

Haripada Datta.

When the series

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots$$

is converted into a continued fraction of the form

$$\frac{a_1}{x+b_1} + \frac{a_2}{x+b_2} + \frac{a_3}{x+b_3} + \dots \qquad \dots (1)$$

we know that the elements of this continued fraction are

$$a_1 = a_1,$$

$$a_n = \frac{K_n K_{n-2}}{K_{n-1}^2},$$

and

$$b_n = \frac{{}^1K_{n-1}}{K_{n-1}} - \frac{{}^1K_n}{K_n},$$

where

$$K_{n+1} = \begin{vmatrix} a_{n+1} & a_n & \dots & a_2 & a_1 \\ a_{n+2} & a_{n+1} & \dots & a_3 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{2n} & a_{2n-1} & \dots & a_{n+1} & a_n \\ a_{2n+1} & a_{2n} & \dots & a_{n+2} & a_{n+1} \end{vmatrix}$$

By deleting the $(r+1)$th row and the last column we obtain ${}^rK_n$.

[1] *This paper was written on the 16th of October 1917, when I was working in Cambridge as a Carnegi Research Student.*

1. With the help of the above theorem the series

$$\frac{a_1}{x} + \frac{a_3}{x^3} + \frac{a_7}{x^7} + \ldots + \frac{a_{2^n-1}}{x^{2^n-1}} + \qquad \ldots (2)$$

is to be converted into a continued fraction.

As in the series (2)

$$a_2 = a_4 = a_5 = a_6 = a_8 = a_9 = \ldots = 0$$

and it can be shown that

$$\left.\begin{aligned}
K_{2^n} &= -a_{2^{n+1}-1} K_{2^n-1} \\
K_{2^n+1} &= a^3_{2^{n+1}-1} K_{2^n-2} \\
K_{2^n+2} &= -a^5_{2^{n+1}-1} K_{2^n-3} \\
&\ldots \quad \ldots \quad \ldots \\
K_{2^{n+1}-2} &= -a^{2^{n+1}-3}_{2^{n+1}-1} K_1 \\
K_{2^{n+1}-1} &= -a^{2^{n+1}-1}_{2^{n+1}-1} K_0
\end{aligned}\right\}$$

where $K_n = 1$, $n = 1, 2, 3, \ldots$ and $2^n + s \neq 2^{n+1}-1$.

From these we can find the a's in terms of a's. All the b's are zeros, for all the determinants of the form $|K_n|$ are zeros.

Making all the a's except a_1 negative we obtain

$$\frac{a_1}{x} - \frac{a_3}{x^3} - \frac{a_7}{x^7} - \ldots$$

$$= \frac{a_1}{x} + \frac{c_2}{x} + \frac{c_3}{-x} + \frac{c_4}{x} + \frac{c_5}{x} + \frac{c_6}{-x} + \frac{c_7}{-x} + \ldots, \qquad \ldots (3)$$

where the 2^nth x, ($n=1, 2, \ldots$) that is the denominator of the 2^n part-fraction or link in the chain, of the continued fraction (3) is positive; and the (2^n-1) x's on the right of the central x are opposite in signs to the corresponding x's on the left of it. All the c's are

positive. The (2^n-2) c's on the right of c_{2^n+1} are equal to the corresponding c's on the left of c_{2^n}; and $c_{2^n} = c_{2^n+1}$. Thus if we know the denominators and the numerators of the first two part-fractions and also the numerators of the 2^nth part-fraction ($n=1, 2, \ldots$) then we can write down the whole continued fraction (3).

As $c_2 = a_3/a_1$; and $c_{2^n} = a_1 a_{2^{n+1}-1} \Big| a_{2^n-1}^2$.

Therefore

$$\frac{a_1}{x} - \frac{a_3}{x^3} - \frac{a_7}{x^7} - \ldots$$

$$= \frac{a_1}{x} + \frac{a_3/a_1}{x} + \frac{a_3/a_1}{-x} + \frac{a_1 a_7/a_3^2}{x} + \frac{a_3/a_1}{-x} + \frac{a_3 a_1}{-x} + \ldots \quad (4)$$

If we consider the convergents in connection with the series and express the denominators as polynomials in x whose coefficients are the functions of the coefficients in the series then it will appear that if the n terms next to the nth term of the power-series

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots$$

are absent then the nth convergent will be a finite series, that is, the first n terms of the series. Thus in the case of the continued fraction (4) the (2^n-1)th convergents are all finite series. In the case of the continued fraction (4) this can be shown, without the intervention of the series, in the following way.

3. Let

$$D_7 = \begin{vmatrix} b_1 & a_2 & & & & & & \\ 1 & b_2 & a_3 & & & & & \\ & 1 & b_3 & a_4 & & & & \\ & & 1 & 0 & a_4 & & & \\ & & & 1 & -b_3 & a_3 & & \\ & & & & 1 & -b_2 & a_2 & \\ & & & & & 1 & -b_1 & \end{vmatrix} = (-1)^7 D_7.$$

Therefore $$D_7 = 0.$$

If the zero in the centre of the continuant be replaced by x_1, and this new continuant be denoted by N_7, then it can be shown that

$$N_7 = D_7 + (-x_1)^4 \begin{vmatrix} b_1 & a_2 & & -b_3 & a_3 & \\ 1 & b_2 & a_3 & 1 & -b_2 & a_2 \\ & 1 & b_3 & & 1 & -b_1 \end{vmatrix}$$

and therefore $$= -x_1 \begin{vmatrix} b_1 & a_2 & \\ 1 & b_2 & a_3 \\ & 1 & b_3 \end{vmatrix}^2$$

If $b_3 = -b_1$; $b_2 = x_2$, and $a_2 = a_3$, then $N_7 = -x_1 x_2^2 b_1^2$.

Thus we see that the denominator of the (2^n-1)th convergent to the continued fraction (4)

$$= -x \cdot x^2 \cdot x^4 \ldots x^{2^{n-1}} = -x^{2^n-1}.$$

Hence the (2^n-1)th convergent is a finite series consisting of the first (2^n-1) terms (zero-coefficients being included) of the series.

4. When a finite series is converted into a continued fraction of the form (1) as well as into another form of continued fraction known as the Equivalent continued fraction, both these continued fractions are finite and equal to each other. Therefore there must be some method or methods of transforming one form of continued fraction into the other. In the case of the continued fraction (4), the transformation can be effected, owing to the presence of the zero-coefficients in the series, in the following way :—

Let us take the continued fraction of the first (2^n-1) part fractions. Leaving aside the first part fraction, if we contract the rest of this finite continued fraction, we obtain a continued fraction of 2^{n-1} part—fractions. Again leaving aside the first two links of this transformed continued fraction and contracting the rest of it, we obtain a continued fraction of $2^{n-2}+1$ links. In this way the continued fraction can be reduced to n part-fractions and which, as a matter of fact, is the equivalent continued fraction, As for example, the seventh convergent to the continued fraction (4), when thus contracted becomes

$$\cfrac{a_1}{x + \cfrac{a_3 x}{a_1 x^2 - a_3 - \cfrac{a_1 a_7 x^2}{a_3 x^4 + a_7}}}.$$

Thus we see though the continued fraction (4) belongs to the form (1), yet it can be regarded as an extended form of the equivalent continued fraction. It is to be noted also that the convergents other than the (2^n-1) th convergent ($n=1, 2\ldots$) are not finite series.

5. As a general case of (2) we can show that

$$\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\ldots+\frac{a_r}{x^r}+\frac{a_{2r+1}}{x^{2r+1}}+\frac{a_{4r+3}}{x^{4r+3}}+\frac{a_{8r+7}}{x^{8r+7}}+\ldots$$

$$=\cfrac{a_1}{x+b_1+}\ \cfrac{a_2}{x+b_2+}\ \cfrac{a_3}{x+b_3+\ldots}\ \cfrac{a_r}{+x+b_r+\ldots}\ \cfrac{a_{r+1}}{x+}\ \cfrac{a_{r+1}}{-(x+b_r)+}$$

$$\cfrac{a_r}{-(x+b_{r-1})+\ldots}\ \cfrac{a_2}{-(x+b_1)+}\ \cfrac{a_{2r+2}}{x+}\ \cfrac{a_{2r+2}}{x+b_1+}\ \cfrac{a_2}{x+b_2+\ldots}$$

$$\ldots\cfrac{a_{r+1}}{x+b_{r-1}+}\ \cfrac{a_{r+1}}{x+b_r+}\ \cfrac{a_r}{-x+}\ \cfrac{a_{r-1}}{-(r+b_r)+\ldots}\ \cfrac{a_2}{-(x+b_1)+}$$

$$\cfrac{a_{4r+4}}{x+}\ \cfrac{a_{4r+4}}{x+b_1+}\ \cfrac{a_2}{x+b_2+\ldots}$$

Here any two quantities $a_2, a_3, \ldots a_r$ are not necessarily equal to one another, and also $b_1, b_2, \ldots b_r$ are not necessarily zeros. If we know the denominators and the numerators of the first r part—fractions as well as the numeratior of the $2^n(r+1)$ th part fraction ($n=0, 1, 2\ldots$) then we can write down the whole of this continued fraction. As for example

$$\frac{1}{x}-\frac{2}{x^2}-\frac{3}{x^3}-\frac{a}{x^7}-\frac{\beta}{x^{15}}-\ldots$$

$$=\cfrac{1}{x-2+}\ \cfrac{1}{x-4+}\ \cfrac{27}{x+6+}\ \cfrac{a/27}{-(x+6)+}\ \cfrac{27}{4-x+}\ \cfrac{1}{2-x+}$$

$$\cfrac{\beta/a^2}{x+}\ \cfrac{\beta/a^2}{x-2+}\ \cfrac{1}{x-4+}\ \cfrac{27}{x+6+}\ \cfrac{a/27}{-x+}\ \cfrac{27}{4-x+}\ \cfrac{1}{2-x+\ldots}$$

6. The series

$$\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\ldots$$

may also be converted into a continued fraction of the form

$$\frac{c_1}{x+}\ \frac{c_2}{1+}\ \frac{c_3}{x+}\ \frac{c_4}{1-} \qquad \ldots \quad (5)$$

where $$c_{2n}=-\frac{K_{n-1}}{K_n}\ \frac{{}^nK_n}{{}^{n-1}K_{n-1}}$$

$$c_{2n+1}=\frac{K_{n+1}}{K_n}\ \frac{{}^{n-1}K_{n-1}}{{}^nK_n}$$

The continued fraction (5) is the extended from of (1), for the $2n$ th convergent of (5) is the same as the n th covergent of (1).[1]

If we convert the series (2) into a continued fraction of the form (5), we find that some of the determinants of the type ${}^nK_n$ vanish, hence some of the quantities c's become zeros or infinities. Though this is the case, *still the even convergents of* (5) *do possess definite values and the continued fraction* (5) *is not meaningless.*

Now we shall enquire into the case where c_{2n} is zero and none of the other c's are zero or infinity. c_{2n} can become zero if either K_{n-1} or ${}^nK_n$ vanishes. In this case K_{n-1} cannot vanish therefore ${}^nK_n$ does vanish. As the c's with suffixes greater than $2n$, are neither zeros nor infinities and therefore

$$K_n=K_{n+1}=\ldots=0, \qquad (6)$$

$${}^nK_{n+1}={}^{n+1}K_{n+2}=\ldots=0. \qquad (7)$$

Now we shall show that if (6) and (7) hold then any convergent of order higher than $2n-1$, is equal to $(2n-1)$ th convergent.

7. Let $$K_3=K_4=0,$$

we know that

$$K_8=\begin{vmatrix} a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \\ a_4 & a_5 & a_7 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}^2=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}K_4.$$

[1] *If the series* $\frac{a_1}{y^2}+\frac{a_2}{y^4}+\frac{a_3}{y^6}+\ldots$ *be converted into a continued fraction, of the form* (1), *this continued fraction owing to the absence of the odd powers of* $\frac{1}{y}$ *in the series will be the same as the continued fraction* (5).

Therefore

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & a_6 \end{vmatrix} = 0.$$

Hence we obtain

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix} = 0. \quad \ldots \quad (8)$$

If also $K_3 = 0.$

then we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_5 & a_6 & a_7 & a_8 \end{vmatrix} = 0.$$

Hence

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \\ a_5 & a_6 & a_7 & a_8 \end{vmatrix} = 0 \quad \ldots \quad (9)$$

and also from (8) we have

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \end{vmatrix} = 0. \quad \ldots \quad (10)$$

From (9) and (10) we obtain

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} = 0. \quad \ldots \quad (11)$$

Now combining (11) with (8) we find that

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \end{Vmatrix} = 0.$$

Similarly if (6) holds then

$$\begin{Vmatrix} a_1 & a_2 & \cdots \\ a_2 & a_3 & \cdots \\ a_{n+1} & a_{n+2} & \cdots \end{Vmatrix} = 0 \qquad \cdots \quad (12)$$

It has been shown in a previous paper[1] that if (12) holds good the even convergents of order higher than $2n$, are all equal to the $2n$th convergent. All the even cenvergents (5) can be connected by recurrence formulae as has been done in (11), and so also the odd convergents. Similarly it can be shown from (7) that any odd convergent of order higher then $(2n-1)$ is equal to the $(2n-1)$th convergent. Exactly in a similar manner it can be shown that if c_n vanishes (whether it is due to the vanishing of K_r or $'K_r$) and none of the other c's is zero or infinity then the continued fraction will terminate at $(n-1)$th convergent.

8. The series

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_4}{x^4} + \frac{a_8}{x^8} + \frac{a_{16}}{x^{16}} + \cdots + \frac{a_{2^n}}{x^{2^n}} + \cdots$$

is to be converted into a continued fraction of the form (1).

Here we have

$$\left.\begin{aligned} K_{2^n} &= a_{2^n} K_0 \\ K_{2^n-1} &= a_{2^n} K_{2^n-2} \\ K_{2^n-2} &= a_{2^n} K_{2^n-4} \\ &\cdots \\ K_{2^{n-1}+1} &= a_{2^n} K_{2^n-2^{n-1}-1} \end{aligned}\right\}$$

[1] On the Failure of Heilermann's Theorem *Proc. Edinburgh Math. Soc.* Vol. XXXV (Part 2), 1916-17.

$$\left.\begin{aligned}
{}^{1}K_{2^n} &= -a_{2^{n+1}}\, K_{2^{n-1}} \\
{}^{1}K_{2^n-1} &= -a_{2^n}^{2^n-2}\, {}^{1}K_1 \\
{}^{1}K_{2^n-2} &= -a_{2^n}^{2^n-4}\, {}^{1}K_2 \\
\cdots &\qquad \cdots \\
{}^{1}K_{2^n+1} &= -a_{2^n}^{2}\, {}^{1}K_{2^{n-1}-1}.
\end{aligned}\right\}$$

From these we find that the continued fraction is

$$\frac{a_1}{x-\frac{a_2}{a_1}+}\ \frac{a_2}{x+b_2+}\ \frac{a_3}{x+b_3+}\ \frac{a_4}{x+b_4+}\ \frac{a_5}{x+b_5+}$$

$$\frac{a_2}{x+b_3+}\ \frac{a_3}{x+b_2+}\ \frac{a_2}{x+b_8+}\ \frac{a_9}{x+a_9+}\ \cdots$$

where

$$a_2=\left(\frac{a_1a_2}{a_1}\right)^2 \qquad b_4=-\frac{a_2}{a_1}+\frac{a_1a_8}{a_4^2}$$

$$a_3=\left(\frac{a_1a_4}{a_2^2}\right)^2 \qquad b_5=-\left(\frac{a_2}{a_1}+\frac{a_1a_8}{a_4^2}\right)$$

$$a_5=\left(\frac{a_1a_8}{a_4^2}\right)^2 \qquad b_8=-\frac{a_2}{a_1}+\frac{a_1a_{16}}{a_8^2}$$

$$a_9=\left(\frac{a_1a^{16}}{a_8^2}\right) \qquad b_9=-\left(\frac{a_2}{a_1}+\frac{a_1a_{16}}{a_8^2}\right)$$

$$b_2=\frac{a_2}{a_1}+\frac{a_1a_4}{a_2^2} \qquad b_{16}=-\frac{a_2}{a_1}+\frac{a_1a_8^2}{a_{16}^2}$$

$$b_3=\frac{a_2}{a_1}-\frac{a_1a_4}{a_2^2} \qquad b_{17}=-\left(\frac{a_2}{a_1}+\frac{a_1a_{32}}{a_{16}^2}\right).$$

From these the laws existing among the denominators and the numerators as regards their arrangement, may be deduced.

Evaluation of the product matrix in a commutantal product of simple matrices having given nullities.

By

C. E. Cullis.

[*Summary.* The object of this paper is the exact determination of every element of the product matrix in a commutantal product of simple matrices in each of which a given number of horizontal rows, a given number of vertical rows, and a given number of diagonal lines, counting from the base in each case, contain only zero elements. Products of four matrices only are considered, these being sufficiently typical of products of any number of matrices. The final results are embodied in the theorems of Art. 7, and contain as particularisations the most important properties of many special kinds of simple matrices. The other articles define and interpret the notations and the nomenclature, and lead up to the general theorems. Readers to whom the meaning of a 'commutantal product' is not clear may suppose in the first instance that all matrices have their natural types, and then make the obvious generalisations.]

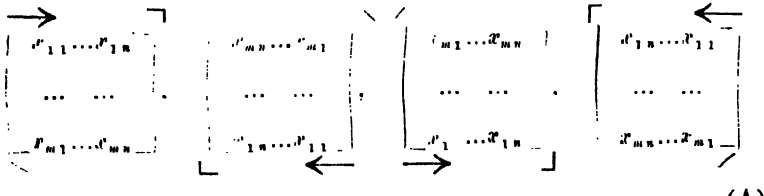
1. Standard notations for simple matrices having assigned types.

1. *Simple matrices having assigned non-ruled types.*

By the four *direct* standard general simple matrices $X=[x]_m^n$ of the class $M\binom{n}{m}$ and of the 1st, 2nd, 3rd, 4th of the non-ruled types

$$\theta_1=\{\pi.\ \pi\},\ \theta_2=\{\pi'.\ \pi'\},\ \theta_3=\{\pi',\ \pi\},\ \theta_4=\{\pi.\ \pi'\} \qquad \ldots\ (a)$$

will be meant the 1st, 2nd, 3rd, 4th of the matrices



$$\ldots\ (A)$$

in which the elements are arbitrary parameters. These are a complete set of simple matrices correlated by simple reversals; and by assigning the type θ_1 to the first of them, we fix the types of the other

three. When we assign a non-ruled type θ to a simple matrix $[x]_m^n$ and say that it is expressed in the standard direct form appropriate to that type, we regard it as a particularisation of the direct standard general simple matrix of the same class and type as defined above, the same notations being used for its elements.

By suitably arranging the conjugates of the four matrices (A) we obtain the four *inverted* standard general simple matrices $X=\ulcorner x \urcorner_n^m$ of the class $M\binom{m}{n}$ and of the 1st, 2nd, 3rd, 4th of the types (a), which we define to be the 1st, 2nd, 3rd, 4th of the matrices

$$\begin{vmatrix} x_{m1}\cdots x_{11} \\ \cdots\ \cdots \\ x_{mn}\cdots x_{1n} \end{vmatrix}, \quad \begin{vmatrix} x_{11}\cdots x_{m1} \\ \cdots\ \cdots \\ x_{1n}\cdots x_{mn} \end{vmatrix}, \quad \begin{vmatrix} x_{m1}\cdots x_{11} \\ \cdots\ \cdots \\ x_{mn}\cdots x_{1n} \end{vmatrix}, \quad \begin{vmatrix} x_{1n}\cdots x_{mn} \\ \cdots\ \cdots \\ x_{11}\cdots x_{m1} \end{vmatrix} \quad \ldots \text{(B)}$$

in which the elements are arbitrary parameters. These are also a complete set of simple matrices correlated by simple reversals; and by assigning the type θ, to the first of them, we fix the types of the other three. When we assign a non-ruled type θ to a simple matrix $\ulcorner x \urcorner_n^m$, and say that it is expressed in the standard inverted form appropriate to that type, we regard it as a particularisation of the inverted standard general simple matrix of the same class and type as defined above, the same notations being used for its elements.

By interchanging m and n in the four matrices (B) we obtain the corresponding four inverted standard general simple matrices $X=\ulcorner x \urcorner_m^n$ of the class $M\binom{n}{m}$. Thus we have defined eight standard general simple matrices, four direct and four inverted, in every class.

In each of the eight matrices (A) and (B), whether general or particularised, we will call

x_{11} the *first* or *initial parametric element*,

x_{mn} the *last* or *final parametric element*,

x_{1n} the *apical element*, and the corner occupied by x_{1n} the *apex*,

x_{m1} the *basical element*, and the corner occupied by x_{m1} the *base*.

The initial element is marked by an arrow, the apex by an arrow head, and the base by a slanting line, which (see Art. 2) is parallel to the parametric diagonal lines, whilst the final element is left unmarked. In each of the four matrices (A) having the direct forms the initial element lies in the same horizontal line as the apex, and in each of the four matrices (B) having the inverted forms it lies in the same vertical line as the apex. The one arbitrary choice of type in (B) is so made that in every standard general simple matrix the positions of the apex and base are fixed by the type. There are eight ways of locating the initial, apical, final and basical elements of a simple matrix when these lie at the four corners, and the apex and base are at opposite corners; and we have eight standard notations for the matrix, one corresponding to each of the eight allocations. They are obtained by supposing the corner elements to carry their names with them both when the order of arrangement of the horizontal or vertical rows is reversed, and when horizontal and vertical rows are interchanged. We can regard (A) and (B) together as forming a complete set of simple matrices correlated by the operations of

reversing the order of arrangement of the horizontal rows,

reversing the order of arrangement of the vertical rows,

replacing a matrix by its conjugate.

A simple matrix of assigned type can be regarded as being built up with

(*i*) successive horizontal rows starting from the base,

(*ii*) successive vertical rows starting from the base,

(*iii*) successive diagonal lines starting from the base.

The standard notations will be helpful in considering these three aspects of the matrix simultaneously; and they will also be used to define special kinds of simple matrices in which the type is a material feature, and cannot be chosen arbitrarily.

NOTE 1. *The conjugate of a standard general simple matrix.*

The conjugates of the four direct standard general simple matrices $X=[x]_m^n$ of the respective types (*a*) are the four inverted standard general simple matrices $X'=\underset{n}{\overset{m}{x}}$ of the respective types

$$\theta_2=\{\pi', \pi'\}, \quad \theta_1=\{\pi, \pi\}, \quad \theta_3=\{\pi', \pi\}, \quad \theta_4=\{\pi, \pi'\}. \qquad \dots, (a')$$

Types represented by correspondingly situated symbolic commutants in the two sets (*a*) and (*a'*) are called *mutually conjugate types*.

NOTE 2. *Transition from one standard form to another.*

Let i, I and j, J be two pairs of positive integers such that

$$i+\mathrm{I}=m+1,\ (i=1,2,\ldots m);\qquad j+\mathrm{J}=n+1,\ (j=1,2,\ldots n);$$

and let $X=[x]^{\theta}_{m}$, $Y=[y]^{\theta}_{m}$ or $X=\underset{\llcorner m}{\overset{\sqcap n}{x}}$, $Y=\underset{\llcorner m}{\overset{\sqcap n}{y}}$

be two standard forms, both direct or both inverted, of the same matrix X having the non-ruled type θ. Then according as the type of Y is

$$\theta,\qquad j\theta j,\qquad j\theta,\qquad \theta j,$$

we have $x_{ij}=y_{ij}$, $x_{ij}=y_{\mathrm{I}\,\mathrm{J}}$, $x_{ij}=y_{\mathrm{I}j}$, $x_{ij}=y_{i\mathrm{J}}$.

Again let $X=[x]^{\theta}_{m}$, $Y=\underset{m}{\overset{\sqcap n}{y}}$ or $X=\underset{\llcorner m}{\overset{\sqcap n}{x}}$, $Y=[y]^{\theta}_{m}$

be two standard forms, one direct and the other inverted, of the same matrix X having the non-ruled type θ. Then according as the type of Y is

$$\theta,\qquad j\theta j,\qquad j\theta,\qquad \theta j,$$

we have $x_{ij}=y_{\mathrm{J}\mathrm{I}}$, $x_{ij}=y_{ji}$, $x_{ij}=y_{\mathrm{J}i}$, $x_{ij}=y_{j\mathrm{I}}$.

These equations provide us with substitutions by which a simple matrix expressed in any one standard form can be expressed in any other standard form. The most useful substitution is $x_{ij}=y_{\mathrm{J}\mathrm{I}}$ which converts a simple matrix expressed in a direct standard form into the inverted standard form of the same type.

NOTE 3. *The natural types.*

The *natural types* are $\{\pi,\pi\}$ when a direct notation is used, and $\{\pi',\pi'\}$ when an inverted notation is used. A simple matrix $[x]^{\theta}_{m}$ to which no type is assigned is ordinarily assumed to have the type $\{\pi,\pi\}$, and to be expressed in the corresponding standard direct form; and a simple matrix $\underset{\llcorner m}{\overset{\sqcap n}{x}}$ to which no type is assigned is ordinarily assumed to have the type $\{\pi',\pi'\}$, and to be expressed in the corresponding standard form. Then $[x]^{\theta}_{m}$ and $\underset{\llcorner n}{\overset{\sqcap m}{x}}$ are mutually conjugate matrices having mutually conjugate non-ruled types.

2. *Parametric elements; horizontal and vertical parametric rows; major-parametric rows.*

In dealing with a simple matrix the use of the term *parametric* always indicates that the matrix is regarded as having an assigned type, and always has reference to the standard general simple matrix (direct or inverted) of the same class and type of which the given matrix is a particularisation. Moreover it serves to indicate an order of counting depending on the type, and to render terms applicable to a simple matrix of natural type also applicable to any simple matrix having an assigned type.

If $X=[x]_{m}^{n}$ or $X=\overset{\ulcorner\ \urcorner}{x}{}_{\llcorner\ \lrcorner}^{m\ n}$ is a given simple matrix having an assigned non-ruled type, then by the *parametric element* x_{ij} of X we mean that element which would be denoted by x_{ij} in the corresponding standard general simple matrix, direct or inverted, of the same class and type. In each of the eight matrices (A) and (B), whether general or particularised, the *pth horizontal parametric row* is the *pth* horizontal row counting from the horizontal row through the initial parametric element x_{11}; the *qth vertical parametric row* is the *qth* vertical row counting from the vertical row through x_{11}; the *qth parametric element* of the *pth* horizontal parametric row is the element common to that row and the *qth* vertical parametric row; the *pth parametric element* of the *qth* vertical parametric row is the element common to that row and the *pth* horizontal parametric row. When the direct notation is used, the parametric element x_{ij} is common to the *ith* horizontal and *jth* vertical parametric rows; when the inverted notation is used, it is common to the *jth* horizontal and *ith* vertical parametric rows.

When we wish to emphasise or make clear the fact that the term parametric is being used in the way just described with reference to a standard general simple matrix, we will replace it by the term *major-parametric*. When a simple matrix of a special kind is regarded as a particularisation of a most general simple matrix of that special kind, it will be natural to use the term parametric in other ways.

3. *Simple matrices having assigned ruled types.*

The 16 ruled types, which include the 4 non-ruled types are represented by the 16 symbolic commutants

$$\theta_{11}=\{\pi,\pi\},\quad \theta_{12}=\{-\pi,-\pi\},\quad \theta_{13}=\{-\pi,\pi\},\quad \theta_{14}=\{\pi,-\pi\}\ ;\quad (a_1)$$
$$\theta_{21}=\{\pi',\pi'\},\quad \theta_{22}=\{-\pi',-\pi'\},\quad \theta_{23}=\{-\pi',\pi'\},\quad \theta_{24}=\{\pi',-\pi'\}\ ;\quad (a_2)$$
$$\theta_{31}=\{\pi',\pi\},\quad \theta_{32}=\{-\pi',-\pi\},\quad \theta_{33}=\{-\pi',\pi\},\quad \theta_{34}=\{\pi',-\pi\}\ ;\quad (a_3)$$
$$\theta_{41}=\{\pi,\pi'\},\quad \theta_{42}=\{-\pi,-\pi'\},\quad \theta_{43}=\{-\pi,\pi'\},\quad \theta_{44}=\{\pi,-\pi'\}\ ;\quad (a_4)$$

where $\theta_{11}=\theta_1$. $\theta_{21}=\theta_2$. $\theta_{33}=\theta_3$. $\theta_{44}=\theta_4$.

In each of these four sets the first type is the *non-ruled equivalent* of all the four types; and a simple matrix to which any one of those four types is ascribed will be said to have the first of them as its *non-ruled equivalent type*. If we suppose that

λ is either 1 or -1, μ is either 1 or -1,

the four sets of symbolic commutants can be represented concisely by the four symbols

$$\{\lambda\pi,\mu\pi\}, \quad \{\lambda\pi',\mu\pi'\}, \quad \{\lambda\pi',\mu\pi\}, \quad \{\lambda\pi,\mu\pi'\} \quad \ldots \quad (b)$$

from which they can be derived respectively by putting in succession

$$[\lambda,\mu]=[1,1], \quad [\lambda,\mu]=[-1,-1], \quad [\lambda,\mu]=[-1,1], \quad [\lambda,\mu]=[1,-1]. \quad (1)$$

By the four direct standard general simple matrices $X=[x]_m^n$ of the class M $(_m^n)$ and of the four types.

$$\theta_{n1}, \quad \theta_{n2}, \quad \theta_{n3}, \quad \theta_{n4}. \qquad (n=1, 2, 3, 4), \qquad \ldots \quad (2)$$

we mean the matrices derived from the first of them (which has been defined in sub-article 1) by multiplying all elements of the

1st, 2nd, 3rd,...ith,...major-parametric horizontal rows by

$$1,\lambda,\lambda^2, \ldots\lambda^{i-1}\ldots;$$

1st, 2nd, 3rd, ...jth,...major-parametric vertical rows by

$$1,\mu,\mu^2,\ldots\mu^{j-1},\ldots;$$

where λ and μ receive in succession the four sets of values given by (1), the initial parametric element x_{11} always remaining unaltered. The positions of the initial parametric elements, apex, final parametric element, base are to be the same in all four matrices as in the first, and the horizontal and vertical parametric rows of all four matrices are to be defined in exactly the same ways as those of the first matrix. For each of the four values of n, the four standard simple matrices x of the respective types (2) form a complete set of simple matrices correlated by the operations of :

changing the signs of all elements in every alternate major-parametric horizontal row, starting with the second such row; ... (3)

changing the signs of all elements in every alternate major-parametric vertical row, starting with the second such row. ... (4)

Again for each of the four values of c the four standard simple matrices X of the respective type

$$\theta_{1c}, \quad \theta_{2c}, \quad \theta_{3c}, \quad \theta_{4c}, \qquad (c=1, 2, 3, 4). \qquad \ldots \quad (2')$$

which are thus defined form a complete set of simple matrices correlated by the operation of :

reversing the order of arrangement of the vertical rows ; ... (3')

reversing the order of arrangement of the horizontal rows ; ... (4')

i.e. correlated by simple reversals ; and the 16 standard simple matrices X which are thus defined form a complete set of simple matrices correlated by the operations (3), (4) and (3'), (4'). These 16

matrices are the 16 *direct standard general simple matrices* $X=[x]^{\ }_{m}$ of the class $M\binom{n}{m}$ and of the 16 ruled types which have been assigned to them. When we ascribe the type θ, non-ruled or ruled, to a simple matrix $[x]^{\ }_{m}$ of the class $M\binom{n}{m}$, and say that it is expressed in the standard direct form appropriate to the type θ, we regard it as a particularisation of the direct standard general simple matrix of the same class and of the type θ, the same notations being used for its elements.

The 16 inverted standard general matrices $X=\underset{\llcorner\ \lrcorner m}{\overset{\sqcap\ \sqcap n}{x}}$ of the 16 ruled types are derived in exactly the same way from the 4 inverted standard general simple matrices $X=\underset{\llcorner\ \lrcorner n}{\overset{\sqcap\ \sqcap m}{x}}$ of the 4 non-ruled types defined in sub-article 1; and the same remarks can be made respecting them. They form a complete set of simple matrices correlated by the operations (3), (4) and (3'), (4'). When we assign the type θ, non-ruled or ruled, to a simple matrix $\underset{\llcorner\ \lrcorner n}{\overset{\sqcap\ \sqcap m}{x}}$ of the class $M\binom{m}{n}$, and say that it is expressed in the standard inverted form appropriate to the type θ, we regard it as a particularisation of the inverted standard general simple matrix of the same class and of the type θ, the same notations being used for its elements.

We have now defined 32 standard general simple matrices in every class, *viz.* 16 direct of the 16 ruled types, and 16 inverted of the 16 ruled types; and we have 32 standard notations for the elements of any given simple matrix. We can however confine ourselves to the use of the direct notations; for we can always pass from the inverted notations to the direct notations. In dealing with special kinds of simple matrices, the inverted notations are often more convenient; but in the present paper we shall generlly use the direct notations. The ruled types, as distinguished from the non-ruled types, are only required when we are concerned with ruled simple matrices, *i.e.*, simple matrices in which the elements of each parametric diagonal line are either all equal or differ only in having signs which are alternately positive and negative.

If $X=[x]^{\ }_{m}$ is a direct standard general simple matrix of any one of the types (*b*), the element common to its i th horizontal and j th

vertical major-parametric rows is

$$\pm x_{i\,j}=\lambda^{i-1}\mu^{j-1}.x_{i\,j}, \qquad (\lambda=\pm 1,\ \mu=\pm 1)\ ; \qquad \ldots\ (5)$$

where $x_{i\,j}$ is an arbitrary parameter which is the same for all the 16 types, and which will be called the parameter of that element. The same remarks apply when $X=\underset{n}{x}^{m}$ is an inverted standard general simple matrix of any one of the types (b), except that we must replace (5) by

$$\pm x_{j\,i}=\lambda^{i-1}\mu^{j-1}.x_{j\,i}, \qquad (\lambda=\pm 1,\ \mu\pm 1). \qquad (5')$$

NOTE 4. *The conjugate of a standard general simple matrix.*

The conjugates of the direct standard general simple matrices $X=[x]^{n}_{m}$ of the class $M\binom{n}{m}$ and of the respective type

$$\{\lambda\pi,\ \mu\pi\}, \qquad \{\lambda\pi',\ \mu\pi'\}, \qquad \{\lambda\pi',\ \mu\pi\}, \qquad \{\lambda\pi,\ \mu\pi'\} \qquad \ldots\ (b)$$

are the inverted standard general simple matrices $X'=\underset{n}{x}^{m}$ of the class $\mathbf{M}\binom{n}{m}$ and of the respective types

$$\{\mu\pi',\ \lambda\pi'\}, \qquad \{\mu\pi,\ \lambda\pi\}, \qquad \{\mu\pi',\ \lambda\pi\}, \qquad \{\mu\pi,\ \lambda\pi'\}. \qquad \ldots\ (b')$$

Types represented by correspondingly situated symbolic commutants in the two sets (b) and (b') are *mutually conjugate types.*

NOTE 5. *Transition from the direct to the inverted notation.*

Let θ be any one of the types (b). Then if $X=[x]^{n}_{m}$ is a simple matrix expressed in the standard direct form appropriate to the type θ and if $Y=\underset{m}{y}^{n}$ is the same matrix expressed in the standard inverted form appropriate to the same type θ, we have

$$\lambda^{i-1}\mu^{j-1}.x_{i\,j}=\lambda^{I-1}\mu^{J-1}.y_{J\,I}, \quad \text{or} \quad x_{i\,j}=\lambda^{m+1}\mu^{n+1}.y_{J\,I},$$

where $\qquad i+I=m+1, \qquad J+J=n+1.$

These equations give us substitution by which we can pass form the direct to the inverted form or from the inverted to the direct form without change of type.

4. *Commutantal products of simple matrices ; commutantal domains.*

Let
$$ABC\ldots K\ L=X \qquad \ldots\ (6)$$

be a commutantal product of given simple matrices A,B,C,...K,L each of which has an assigned type ; let that type be assigned to X which renders the equation (6) commutantal, *i.e.* let the symbolic commutant which represents the type of X be the product of the symbolic commutants which represent the types of A, B, C,...K, L ; and let A, B, C,...K, L, X be expressed in the standard direct forms appropriate to

the types assigned to them. Then the expressions for the parameters of X in terms of the parameters of A, B, C, ..K, L are independent of the type of the product; and in finding them we can always replace (6) by any correlated commutantal product

$$A'B'C'\ldots K'L'=X' \qquad \ldots \quad (6')$$

We usually chose (6') so that A', B', C',...K',L' all have the ordinary type $\{\pi, \pi\}$.

The same remarks are applicable when A, B, C,...K, L are all expressed in the standard inverted forms appropriate to their types; and we may then chose (6') so that A', B', C',...K', L' all have the ordinary type $\{\pi', \pi'\}$.

Thus in evaluating the product matrix in any commutantal product of t factor matrices, we are simultaneously evaluating the product matrices in all the 4^{t-1} correlated commutantal products.

All simple matrices of a kind S to which commutantal types can be assigned from a *commutantal domain* when in every commutantal product such as (6) in which A, B, C,...K, L are simple matrices of the kind S having assigned types the product matrix X is always a simple matrix of the same kind S and of the type which renders the equation (6) commutantal.

2. The parametric diagonal lines of a simple matrix.

Let $X=[\cdot]_m^n$ or $X=[x]_n^m$ be any simple matrix which has an assigned type and is expressed in the standard direct or inverted form appropriate to that type. The parametric element $\pm x_{ij}$ or its parameter x_{ij} will be said to have

$$\text{difference-weight } j-i, \qquad \text{total-weight } i+j.$$

The matrix X contains two sets of diagonal lines, *viz.*

(1) $m+n-1$ mutually parallel diagonal lines each having the property that all the elements $\pm x_{ij}$ lying on it have the same difference-weight $j-i$, the difference-weights of the successive lines starting from the base being

$$1-m, \quad 2-m, \ldots \quad n-2, \quad n-1;$$

(2) $m+n-1$ mutually parallel diagonal lines each having the property that all the elements $\pm x_{ij}$ lying on it have the same

total weight $i+j$, the total weights of the successive lines starting from the initial parametric element being

$$2, 3, \ldots m+n-1, m+n.$$

We will lay the greater stress on the diagonal lines of constant difference-weight, and call these the *parametric diagonal lines*. They will ordinarily be counted from the base to the apex. The first or initial parametric element of any such line is that element $\pm x_{ij}$ lying on it for which i and j have the smallest values.

In any simple matrix those of the diagonal lines which slope downwards towards the right, or are parallel to the leading diagonal, are the *ante-diagonal lines*; those of them which slope downwards towards the left are the *counter-diagonal lines*. The diagonal lines drawn through the matrix from the four corners are the *diagonals* of the matrix, the diagonal drawn through the matrix from the top left-hand corner being the *leading diagonal*. When the matrix is not square, there are four diagonals, two of them being *ante-diagonals* and two being *counter-diagonals*; when the matrix is square, there are only two diagonals, *viz.* one ante-diagonal and one counter-diagonal. The leading diagonal is always an ante-diagonal. The meanings of these terms are independent of the type of the matrix and of the form in which it is expressed.

The diagonal lines of constant total weight drawn through the matrix from the apex and base (which become one when the matrix is square) will be called respectively the *apical diagonal* (or *median line*) and the *basical diagonal* of the matrix. The parametric diagonal lines (of constant difference-weight) drawn through the matrix from the initial and final parametric elements are the *parametric diagonals* (there being only one parametric diagonal when the matrix is square). We will distinguish between the two parametric diagonals by calling one of them the pre-major and the other the post-major diagonal. The *pre-major diagonal* is drawn through the matrix from the other corner lying in the same horizontal line as the apex (or the same vertical line as the base); the *post-major diagonal* is drawn through the matrix from the other corner lying in the same vertical line as the apex (or the same horizontal line as the base); the *non-major diagonal* is that one of the two parametric diagonals which lies nearer to the apex (or more remote from the base). The parametric diagonal lines extending from the pre-major diagonal to the apex, from the post-major diagonal to the apex, and

from the non-major diagonal to the apex will be called respectively the *pre-major diagonal lines*, the *post-major diagonal lines*, and the *non-major diagonal lines*. The pre-major or post-major or non-major diagonal is the first pre-major or first post-major or first non-major diagonal line. The meanings of all these terms depend on and are fixed by the type; when the type or only its non-ruled equivalent is given, so that the positions of the apex and base are fixed, the meanings are independent of the form in which the matrix is expressed. The words 'pre-major' and 'post-major' have however been coined (See Exs. *i* and *ii*) with special reference to the direct standard forms.

In a simple matrix expressed in the standard form, direct or inverted, appropriate to one of the 8 types $\{\pm\pi, \pm\pi\}$, $\{\pm\pi', \pm\pi'\}$ the parametric diagonal lines are the ante-diagonal lines; and these types are therefore called the *ante-types*. In a simple matrix expressed in the standard from, direct or inverted, appropriate to one of the 8 types $\{\pm\pi', \pm\pi\}$, $\{\pi, \pm\pi'\}$ the parametric diagonal lines are the counter-diagonal lines; and these types are therefore called the *counter-types*. The conjugate of an ante-type is always an ante-type; and the conjugate of a counter-type is always a counter-type.

Ex. i If $+x_{ij}$ is a parametric element of a simple matrix $X=[x]_{m}^{n}$ or $X=\overset{n}{\underset{m}{x}}$ expressed in the standard direct or inverted form appropriate to its type, the possible values of i and j, the possible values of the difference-weight $k=j-i$, and the possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i \not< 1,\ i \not> m,\ j \not< 1,\ j \not> n; \qquad \ldots \quad (1)$$

$$k \not< 1-m,\ k \not> n-1; \qquad \ldots \quad (1')$$

$$i \not< 1,\ i \not< 1-k,\ i \not> n-k,\ i \not> m;\ j \not< 1,\ j \not< 1+k,\ j \not> m+k,\ j \not> n \qquad \ldots \quad (1'')$$

i and j being horizontal and vertical indicators in $[x]_{m}^{n}$, and vertical and horizontal indicators in $\overset{n}{\underset{m}{x}}$. The two sets of inequalities (1'') show the horizontal and vertical or the vertical and horizontal parametric rows in which the parametric elements of difference-weight k lie.

The pre-major and post-major diagonals pass respectively through:

the initial element x_{11} and the final element $\pm x_{mn}$ in $[x]_{m}^{n}$;

the final element $\pm x_{mn}$ and the initial element x_{11} in $\overset{n}{\underset{m}{x}}$.

Ex. ii If $\pm y_{ji}$ is a parametric element of simple matrix $Y=\overset{n}{\underset{m}{y}}$ or $Y=[y]_{m}^{n}$ expressed in the standard inverted or direct standard form appropriate to its type,

the possible values of the difference-weight $k=i-j$, and the possible values of j and i when k is given are those consistent with the respective sets of conditions

$$j \nless 1,\ j \ngtr n,\ i \nless 1,\ i \ngtr m\ ; \qquad \ldots\ (2)$$

$$k \nless 1-n,\ k \ngtr m-1\ ; \qquad \ldots\ (2')$$

$$j \nless 1,\ j \nless 1-k,\ j \ngtr m-k,\ j \ngtr n\ ;\ i \nless 1,\ i \nless 1+k,\ i \ngtr n+k,\ i \ngtr m\ ; \qquad \ldots\ (2'')$$

j and i being vertical and horizontal indicators in $\underset{\_}{\overset{\frown}{y}}{}^{n}_{m}$, and horizontal and vertical indicators in $[y]^{n}_{m}$.

The pre-major and post-major diagonals pass respectively through :

the final element $\pm y_{m\,n}$ and the initial element $y_{1\,1}$ in $\underset{\_}{\overset{\frown}{y}}{}^{n}_{m}$;

the initial element $y_{1\,1}$ and the final element $\pm y_{m\,n}$ in $[y]^{m}_{n}$.

3. The nullities and the parametric rank of a simple matrix of given type.

1. *The basical nullity and the parametric (or apical) rank.*

Let $X=[x]^{n}_{m}$ or $X=\underset{\_}{\overset{\frown}{x}}{}^{m}_{n}$ be any given simple matrix of the class $M\binom{n}{m}$ or $M\binom{m}{n}$ which has an assigned type and is expressed in the standard direct or inverted form appropriate to the type.

We will define the *basical nullity* ξ of X to be the exact number of consecutive parametric diagonal lines counting from the base which contain only zero elements, the possible values of ξ being given by

$$\xi \nless 0,\ \xi \ngtr m+n-1.$$

If X is a zero matrix, we have $\xi = m+n-1$. If X is not a zero matrix, so that $\xi < m+n-1$, the first ξ parametric diagonal lines of X counting from the base contain only zero elements, but the next parametric diagonal line, which will be called the *first effective parametric diagonal line*, contains at least one non-zero element. In this latter case all elements of X having difference-weights not greater than $\xi - m$ are equal to 0, the least difference-weight of a non-zero element being $\xi - m+1$; and the parametric diagonal lines extending from the first effective parametric diagonal line to the apex will be called the *effective parametric diagonal lines*, their difference-weights being

$$\xi - m+1,\ \xi - m+2, \ldots n-2,\ n-1.$$

The last definition remains valid even when some or all of the parametric diagonal lines coming after the first effective parametric diagonal line contain only zero elements.

In place of the basical nullity ξ it is generally more convenient to use the *parametric* (or *apical*) rank ξ, which we define to be the total number of the effective parametric diagonal lines. It is the positive integer ξ determined by the equation

$$\xi+\bar{\xi}=m+n-1, \qquad \ldots \quad (1)$$

the possible values of which are those consistent with the conditions

$$\xi \not< 0,\ \xi \not> m+n-1.$$

Because of the relation (1) we can always put

$$\xi=n-\xi_n,\ \bar{\xi}=m-1+\xi_n, \qquad \ldots \quad (2)$$

where ξ_n is a uniquely determinate integer which is subject to the conditions

$$\xi_n \not< 1-m,\ \xi_n \not> n.$$

When $\xi_n=n$, X is a zero matrix; if $\xi_n<n$, then ξ_n is the least difference-weight of a non-zero element of X, *i.e.* the difference-weight of the first effective parametric diagonal line. In all cases we will call ξ_n the *least effective difference-weight* of X, interpreting it to be n when X is a zero matrix.

The excess of the total number of effective parametric diagonal lines over the total number of pre-major parametric diagonal lines (or vertical rows) will be called the *pre-major parametric excess*. It is an integer which may be either positive or negative, and when its sign is changed, it becomes the pre-major parametric *defect*. The pre-major parametric excess is

$$\xi-n \text{ when } X=\left[\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}\right]_n^m, \quad \xi-m \text{ when } X=\left[\begin{smallmatrix} \cdot & \cdot \end{smallmatrix}\right]_n^m.$$

The excess of the total number of effective parametric diagonal lines over the total number of post-major parametric diagonal lines (or horizontal rows) will be called the *post-major parametric excess*. It is an integer which may be either positive or negative; and when its sign is changed, it becomes the post-major parametric *defect*. The post-major parametric excess is

$$\xi-m \text{ when } X=\left[\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}\right]_n^m, \ \xi-n \text{ when } X=\left[\begin{smallmatrix} \cdot & \cdot \end{smallmatrix}\right]_n^m.$$

2. *The horizontal and vertical nullities.*

The *horizontal nullity* of a simple matrix having an assigned type is the total number of consecutive rows, counting from the horizontal row through the base, which contain only zero elements; consequently it is the total number of

final horizontal or initial horizontal

zero major-parametric rows according as the direct or inverted standard notation appropriate to the type is used.

The *vertical nullity* of a simple matrix having an assigned type is the total number of consecutive vertical rows, counting from the vertical row through the base, which contain only zero elements; consequently it is the total number of

initial vertical or final vertical

zero major-parametric rows according as the direct or inverted standard notation appropriate to the type is used.

When the direct forms are regarded as fundamental, we will denote the horizontal and vertical nullities of a simple matrix X by ξ_1 and ξ_2 or by ξ_2 and ξ_1 according as X is expressed in the direct or inverted standard form appropriate to its type.

First let $X=[e]_m^n$ be a simple matrix of the class $M\binom{u}{m}$ which is expressed in the standard direct form appropriate to its type, and which has

parametric rank ξ, horizontal nullity ξ_1, vertical nullity ξ_2,

where ξ, ξ_1, ξ_2 are essentially positive integers. If X is not a zero matrix, we have

$$\xi>0, \quad \xi_1<m, \quad \xi_2<n,$$

and we can express it in the 1st, 2nd, 3rd, 4th of the forms

$$\begin{bmatrix} 0, x' \\ 0, 0 \end{bmatrix}_{m-\xi_1,\ \xi_1}^{\xi_2,\ n-\xi_2}, \quad \begin{bmatrix} 0, 0 \\ x', 0 \end{bmatrix}_{\xi_1,\ m-\xi_1}^{n-\xi_2,\ \xi_2}, \quad \begin{bmatrix} 0, 0 \\ x', 0 \end{bmatrix}_{\xi_1,\ m-\xi_1}^{\xi_2,\ n-\xi_2},$$

$$\begin{bmatrix} x', 0 \\ 0, 0 \end{bmatrix}_{m-\xi_1,\ \xi_1}^{n-\xi_2,\ \xi_2} \quad \dots \quad (A)$$

according as its type or the non-ruled equivalent of its type is represented by the 1st, 2nd, 3rd, 4th of the symbolic commutants

$$\theta_1=\{\pi, \pi\}, \ \theta_2=\{\pi', \pi'\}, \ \theta_3=\{\pi', \pi\}, \ \theta_4=\{\pi, \pi'\} \quad \dots \quad (a)$$

In each case the constituent $X_n = [x']_{m-\xi_1}^{n-\xi_2}$ of X is a simple matrix having the same non-ruled equivalent type as X, and having

parametric rank ξ, horizontal nullity 0, vertical nullity 0;

and we will call X_n the *least effective constituent* of X. The horizontal and vertical rows of X which occur in X_n will be called the *effective* horizontal and vertical *parametric rows* of X, the $\begin{matrix}\text{last horizontal}\\ \text{first vertical}\end{matrix}$ parametric row of X_n being (when extended) the $\begin{matrix}\text{last effective horizontal}\\ \text{first effective vertical}\end{matrix}$ parametric row of X, *i.e.* the $\begin{matrix}\text{last horizontal}\\ \text{first vertical}\end{matrix}$ parametric row of X which contains a non-zero element. According to previous definitions the effective parametric diagonal lines of X_n are (when extended) the effective parametric diagonal lines of X.

Next let $X = \overset{m}{\underset{n}{[x]}}$ be a simple matrix of the class $M\binom{m}{n}$ which is expressed in the standard inverted forms appropriate to its type and has

parametric rank ξ, horizontal nullity ξ_2, vertical nullity ξ_1,

where ξ, ξ_1, ξ_2 are essentially positive integers. If X is not a zero matrix, we have

$$\xi > 0, \quad \xi_1 < m, \quad \xi_2 < n,$$

and we can express it in the 1st, 2nd, 3rd, 4th of the forms

$$\begin{matrix}\left[\begin{matrix}\xi_1,\ m-\xi_1\\ 0,\ x'\end{matrix}\right. & \left[\begin{matrix}m-\xi_1,\\ 0,\ 0\end{matrix}\right. & \left[\begin{matrix}\xi_1,\ m-\xi_1\\ 0,\ 0\end{matrix}\right. & \left[\begin{matrix}m-\xi_1,\ \xi_2\\ x',\ 0\end{matrix}\right.\\ \left.\begin{matrix}0,\ 0\\ n-\xi_2,\ \xi_1\end{matrix}\right], & \left.\begin{matrix}x',\ 0\\ \xi_2,\ n-\end{matrix}\right], & \left.\begin{matrix}0,\ x'\\ \xi_2,\ n-\xi_2\end{matrix}\right], & \left.\begin{matrix}0,\ 0\\ n-\xi_2,\ \xi_2\end{matrix}\right]\end{matrix} \quad \ldots \text{(B)}$$

according as its type or the non-ruled equivalent of its type is represented by the 1st, 2nd, 3rd, 4th of the symbolic commutants (a).

In each case the constituent $X_n = \left[\begin{matrix}m-\xi_1\\ x'\\ n-\xi_2\end{matrix}\right]$ of X is a simple matrix having the same non-ruled equivalent type as X, and having

parametric rank ξ, horizontal nullity 0, vertical nullity 0;

and X_α is the least effective constituent of X. The horizontal and vertical rows of X which occur in X_α are the *effective* horizontal and vertical *parametric* rows of X, the $\begin{matrix}\text{first horizontal}\\ \text{last vertical}\end{matrix}$ parametric row of X_α being (when extended) the $\begin{matrix}\text{first effective horizontal}\\ \text{last effective vertical}\end{matrix}$ parametric row of X, *i.e.* the $\begin{matrix}\text{first horizontal}\\ \text{last vertical}\end{matrix}$ parametric row of X which contains a non-zero element. The effective parametric diagonal lines of X_α are (when extended) the effective parametric diagonal lines of X. We could of course regard the 1st, 2nd, 3rd, 4th of the matrices (B) and their least effective constituents as the conjugates of the 2nd, 1st, 3rd, 4th of the matrices (A) and their least effective constituents.

When $X=[x]_m^n$ has the natural type $\{\pi,\pi\}$ or $X=\begin{matrix}n\\ x\\ m\end{matrix}$ has the natural type $\{\pi',\pi'\}$, the horizontal nullity is the total number of consecutive final horizontal rows of 0's, and the vertical nullity is the total number of consecutive initial vertical rows of 0's.

Whenever a simple matrix X which is expressed in the standard direct or inverted form appropriate to its type has that one of the forms (A) or (B) which corresponds to its type, the non-zero constituent occupying the corner at the apex, we will call X_α the *effective constituent* of X. But ξ_1 and ξ_2 or ξ_4 and ξ_3 are the horizontal and vertical nullities only when X_α is the *least* effective constituent, *i.e.* when X_α has zero horizontal and vertical nullities, so that ξ_1 and ξ_2 or ξ_3 and ξ_4 have their greatest possible values.

In the following examples X is defined either as in (A) or as in (B), so that in both cases the relations (1) and (2) hold good.

Ex. i. When X is a zero matrix, the least effective constituent X_α is non-existent and we have

$$\xi=0,\quad \xi_1=m,\quad \xi_2=n.$$

Ex. ii. When X is a non-zero matrix having parametric rank ξ and zero horizontal and vertical nullities, the possible values of ξ are those consistent with the conditions

$$\xi\leqslant m,\quad \xi\leqslant n,\quad \xi\geqslant m+n-1.$$

In fact the characteristic property of a non-zero simple matrix having zero horizontal and vertical nullities is that its parametric rank cannot be less than either of its two orders.

Ex. iii. When X is a non-zero simple matrix, the possible simultaneous values of the positive integers ξ, ξ_1, ξ_2 are those consistent with the conditions

$$\xi_1\leqslant 0,\ \xi_2\leqslant 0,\ \xi+\xi_1\leqslant m,\ \xi+\xi_2\leqslant n,\ \xi+\xi_1+\xi_2\geqslant m+n-1;$$

which include the conditions

$$\xi \leqslant 1,\ \xi \geqslant m+n-1\ ;\quad \xi_1 \leqslant 0,\ \xi_1 \geqslant m-1,\ \xi_2 \leqslant 0,\ \xi_2 \geqslant n-1$$

giving respectively the possible values of ξ, and the possible simultaneous values of ξ_1 and ξ_2.

This follows from Ex. iii and the forms (A) or (B) into which X can be put.

Ex. iv. Under the same circumstances the possible simultaneous values of ξ_0, ξ_1, ξ_2 are those consistent with the conditions.

$$\xi_1 \leqslant 0,\ \xi_2 \leqslant 0,\ \xi_0 - \xi_1 \geqslant n-m,\ \xi_0 - \xi_2 \geqslant 0,\ \xi_1 + \xi_2 = \xi_0 \geqslant m-1,$$

which include $$\xi_0 \leqslant 1-m,\ \xi_0 \geqslant n-1\ ;$$

and the possible simultaneous values of $\bar{\xi}, \xi_1, \xi_2$ are those consistent with the conditions

$$\xi_1 \leqslant 0,\ \xi_2 \leqslant 0,\ \bar{\xi} \geqslant m+n-2,\ \bar{\xi} - \xi_1 \geqslant n-1,\ \bar{\xi} - \xi_2 \geqslant m-1,\ \xi_1 + \xi_2 - \bar{\xi} \geqslant 0,$$

which include $$\bar{\xi} \leqslant 0,\quad \bar{\xi} \geqslant m+n-2.$$

In place of the essentially positive integers ξ_1, ξ_2 it will often be more convenient to use the essentially positive integers

$$\xi_{10} = \xi + \xi_1 - m,\quad \xi_{20} = \xi + \xi_2 - n.$$

Then ξ_{10} is the post-major, ξ_{20} the pre-major excess of X,

$$\text{when } \mathrm{X} = \left[\,\right]_m^n$$

and ξ_{10} is the pre-major, ξ_{20} the post-major excess of X,

$$\text{when } \mathrm{X} = \left[\!\!\left[\,\right]\!\!\right]_n^m .$$

The three nullities of X (basical, horizontal, vertical) are known when the values of ξ, ξ_1, ξ_2 or the values of ξ, ξ_{10}, ξ_{20} are given.

Ex. v. When X is a zero matrix, we have

$$\xi = 0,\quad \xi_{10} = 0,\quad \xi_{20} = 0.$$

Ex. vi. When X is a non-zero matrix, the possible simultaneous values of the positive integers ξ, ξ_{10}, ξ_{20} are those consistent with the conditions

$$\xi_{10} \leqslant 0,\ \xi_{10} \leqslant \xi - m,\ \xi_{20} \leqslant 0,\ \xi_{20} \leqslant \xi - n,\ \xi_{10} + \xi_{20} \geqslant \xi - 1,$$

which include the conditions

$$\xi \leqslant 1,\ \xi \geqslant m+n-1\ ;\quad \xi_{10} \leqslant 0,\ \xi_{10} \geqslant n-1,\ \xi_{20} \leqslant 0,\ \xi_{20} \geqslant m-1$$

giving respectively the possible values of ξ, and the possible simultaneous values of ξ_{10} and ξ_{20}.

NOTE.—*Transition from the direct to the inverted forms.*

When the inverted forms are regarded as fundamental, we will denote the horizontal and vertical nullities of a simple matrix Y by η_1 and η_2 or by η_2 and η_1 according as Y is expressed in the inverted or direct standard form appropriate to its type. Then if the typical parametric element of Y is y_{ij} having difference-weight $k=i-j$, we can convert all the formulae relating to $\mathrm{X}=[x]^{m}_{n}$ or $\mathrm{X}=\underset{m}{\overset{n}{x}}$ which have been obtained into the corresponding formulae relating to a simple matrix

$\mathrm{Y}=\underset{n}{\overset{m}{y}}$ having

parametric rank η, horizontal nullity η_1, vertical nullity η_2

or $\mathrm{Y}=[y]^{m}_{n}$ having

parametric rank η, horizontal nullity η_2, vertical nullity η_1

by replacing $m, n ; i, j ; \xi_1, \xi_2 ; \xi_{10}, \xi_{20} ; \xi, \bar{\xi}, \xi_0, k, \mathrm{X}, \mathrm{X}_0$

by $n, m ; j, i ; \eta_2, \eta_1 ; \eta_{20}, \eta_{10} ; \eta, \bar{\eta}, \eta_0, k, \mathrm{Y}, \mathrm{Y}_0$.

We then have $\eta_{10}=\eta+\eta_1-m, \quad \eta_{20}=\eta+\eta_2-n$;

where η_{10} is the post-major, η_{20} the pre-major excess of Y, when $\mathrm{Y}=\underset{n}{\overset{m}{y}}$,

and η_{10} is the pre-major, η_{20} the post-major excess of Y, when $\mathrm{Y}=[y]^{m}_{n}$.

3. *Simple slopes.*

Let parametric diagonal lines be counted from the base. Then a simple matrix of any given class which is expressed in the standard direct or inverted form appropriate to the type θ will be called :

a *simple pre-major slope* of that class and of the type θ when all parametric diagonal lines preceding the pre-major diagonal contain only zero elements,

or the effective parametric diagonal lines are all pre-major diagonal lines,

or the parametric rank does not exceed the number of vertical rows,

or the pre-major parametric excess is not greater than 0 ;

a *simple post-major slope* of that class and of the type θ when all parametric diagonal lines preceding the post-major diagonal contain only zero elements,

or the effective parametric diagonal lines are all post-major diagonal lines,

or the parametric rank does not exceed the number of horizontal rows,

or the post-major parametric excess is not greater than 0;

a *simple non-major slope* of that class and of the type θ when all parametric diagonal lines preceding the non-major diagonal (the parametric diagonal lying nearer to the apex) contain only zero elements,

or the effective parametric diagonal lines are all non-major diagonal lines,

or the parametric rank does not exceed the smaller or effective order of the matrix,

or the pre-major and post-major parametric excesses are both not greater than 0.

Thus a *simple non-major slope* of a given class and type is a matrix which is both a simple pre-major slope and a simple post-major slope of that class and type. By a *simple major slope* of a given class and type will be meant a matrix which is either a simple pre-major slope or a simple post-major slope of that class and type.

The above definitions depend only on the class and the non-ruled equivalent of the type, *i.e.* on the class and the position assigned to the apex, being independent (when these are given) of the form in which the matrix is expressed; but the most general form of a simple pre-major or post-major or non-major slope of a given class and type does depend on the position assigned to the apex, *i.e.* on the non-ruled equivalent of the type. Consequently in every class we have simple pre-major and post-major and non-major slopes:

(*i*) of the type $\{\pi, \pi\}$ to which any one of the four types $\{\pm\pi, \pm\pi\}$ can be ascribed;

(*ii*) of the type $\{\pi', \pi'\}$ to which any one of the four types $\{\pm\pi', \pm\pi'\}$ can be ascribed;

(*iii*) of the type $\{\pi', \pi\}$ to which any one of the four types $\{\pm\pi', \pm\pi\}$ can be ascribed;

(*iv*) of the type $\{\pi, \pi'\}$ to which any one of the four types $\{\pm\pi, \pm\pi'\}$ can be ascribed.

These will be called respectively simple pre-major and post-major and non-major slope of the *first*, *second*, *third*, *fourth* types. Those having the ante-types (*i*) and (*ii*) will be called simple pre-major and

post-major and non-major *ante-slopes*, because in them the parametric diagonal lines are ante-diagonal lines. Those having the counter-types (*iii*) and (*iv*) will be called simple pre-major and post-major and non-major *counter-slopes*, because in them the parametric diagonal lines are counter-diagonal lines.

Ex. vii. The simple matrix $X=[x]_m^n$ to which the type θ is ascribed is :

a simple *pre-major slope* of type θ when $\xi \nsucc n$, or $\xi_\circ \nprec 0$;

a simple *post-major slope* of type θ when $\xi \nsucc m$, or $\xi_\circ \nprec n-m$;

a simple *non-major slope* of type θ when $\xi \nsucc m$, $\xi \nsucc n$; or $\xi_\circ \nprec 0$, $\xi_\circ \nprec n-m$.

The simple $X=\begin{matrix}\ulcorner\!\!\urcorner^m \\ \llcorner\!\!\lrcorner_n\end{matrix} x$ to which the type θ is ascribed is :

a simple *pre-major slope* of type θ when $\xi \nsucc m$, or $\xi_\bullet \nprec n-m$;

a simple *post-major slope* of type θ when $\xi \nsucc n$, or $\xi_\bullet \nprec 0$;

a simple *non-major slope* of type θ when $\xi \nsucc m$, $\xi \nsucc n$; or $\xi_\bullet \nprec 0$, $\xi_\bullet \nprec n-m$.

4. The product matrix in a commutantal product of general simple matrices.

We consider first the commutantal product

$$[a]_m^p \ [b]_p^q \ [c]_q^r \ [d]_q^n = [e]_m^n \quad \text{or} \quad ABCD=X \quad \ldots \quad (A)$$

in which A, B, C, D are four direct general standard simple matrices of given commutantal types (non-ruled or ruled) ; we ascribe to X the type appropriate to the product, so that the symbolic commutant representing the type of X is the product of the successive symbolic commutants representing the types of A, B, C, D ; and we regard X as a simple matrix expressed in the standard direct form appropriate to its type. Then the element common to the i th horizontal and jth vertical parametric rows of X is $\pm x_{i,j}$, where the sign depends on the type of X, and is known ; and X will be completely known when we know all its parameters such as $x_{i,j}$. The matrices A, B, C, D have zero basical, zero horizontal, and zero vertical nullities, and the given parametric (or apical) ranks

$$\alpha=m+p-1,\ \beta=p+q-1,\ \gamma=q+r-1,\ \delta=q+n-1,$$

their parameters being all arbitrary and independent.

If $x_{i,j}$ is a parameter of X, the possible values of i and j, the possible values of the difference-weight $k=j-i$, and possible values

of i and j when k is given are those consistent with the respective sets of conditions

$$i \nless 1,\ i \ngtr m,\ j \nless 1,\ j \ngtr n\ ; \qquad \ldots \quad (1)$$

$$k \nless 1-m,\ k \ngtr n-1\ ; \qquad \ldots \quad (1')$$

$$i \nless 1,\ i \nless 1-k,\ i \ngtr n-k,\ i \ngtr m\ ;\ j \nless 1,\ j \nless 1+k,\ j \ngtr m+k,\ j \ngtr n. \quad \ldots \quad (1'')$$

The two sets of conditions (1″) are mutually equivalent, and show respectively the horizontal and vertical parametric rows of X in which the elements $x_{i,j}$ having a given possible difference-weight k lie; and each of them together with (1′) and the equation $k=j-i$ is equivalent to (1).

There are 4^5 possible types of the commutantal product (A); but in all of them the expression for any given parameter $x_{i,j}$ of X as a rational integral function of the parameters of A, B, C, D is the same, being independent of the type, and in obtaining it we may suppose A, B, C, D, X to all have the same natural type $\{\pi, \pi\}$. Accordingly (see § 51 of *Matrices and Determinoids, Vol. 1*), it is the expression given in the following theorem :

Theorem I. *Every parametric element of the product matrix X in the commutantal product (A) is given by the general formula*

$$x_{i,j}=\Sigma a_{i,u}\, b_{u,v}\, c_{v,w}\, d_{w,j}, \quad (j-i=k), \qquad \ldots \quad (a)$$

where the possible values of i, j and k are given by (1), (1′), (1″), and where for a given pair of values of i and j there are pqr terms in the sum (a) obtained by giving to u, v, w the values

$$u=1, 2, \ldots p\ ; \quad v=1, 2, \ldots q\ ; \quad w=1, 2, \ldots r.$$

Consequently none of the elements of X vanish identically; the basical, horizontal and vertical nullities of X are all equal to 0; and X has the parametric rank ξ given by

$$\xi=m+n-1.$$

The summation in (a) extends over all distinct sets of values of u, v, w which can be formed when u, v, w are elements of the respective sequences

$$[1, 2, \ldots p], \quad [1, 2, \ldots q], \quad [1, 2, \ldots r].$$

The second part of the theorem is a consequence of the fact that the parameters of A, B, C, D are all arbitrary and independent, there being no relations between them. The expression for $x_{i,j}$ is the

same for all possible values of k; but the elements of X having a given difference-weight k lie only in the horizontal and vertical parametric rows of X given by the first and second of the two sets of conditions (1″). It is particularly important to observe that :

The expression (*a*) *for a parameter* x_{ij} *of X having a given difference-weight* $j-i=k$ *is isobaric of weight* k *in the difference-weights of the parameters of A, B, C, D, ; for we have*

$$(u-i)+(v-u)+(w-v)+(j-w)=j-i=k.$$

In every commutantal product of four simple matrices of the classes M $\binom{r}{m}$, M $\binom{s}{p}$, M $\binom{t}{q}$, M $\binom{u}{r}$ the factor matrices are particularisations of A, B, C, D in the commutantal product (A) of the same type, and the product matrix is a particularisation of X. But when A, B, C, D are particularised by assigning particular values to some of their parameters or by introducing relations between the parameters, then in the expression for x_{ij} all the pqr terms may vanish, in which case we certainly have $x_{ij}=0$, or some only of those terms may vanish, in which case the expression for x_{ij} can be simplified by the omission of the zero terms. The following articles deal with certain particularisations of the product (A) and the corresponding simplifications of the formula (*a*).

When A, B, C, D are particularised in any way, so that X also is particularised, let α_0, β_0, γ_0, δ_0, ξ_0 be the least effective difference-weight of A, B, C, D, X. Then if x_{ij} is any element of X having difference-weight $k=j-i$, and if there exists a non-zero term $a_{iu}\, b_{vr}\, c_{tw}\, d_{uj}$ in the expression (*a*) for x_{ij}, we must have

$$u-i \not< \alpha_0,\ v-u \not< \beta_0,\ w-r \not< \gamma_0,\ j-w \not< \delta_0,$$

and therefore

$$k \not< \alpha_0+\beta_0+\gamma_0+\delta_0. \qquad \ldots \quad (2)$$

Consequently we have the following general theorem relating to all such particularisations :

Theorem II. *Whenever in any particularisation of the product* (*A*) *the product matrix X is not a zero matrix, its least effective difference-weight* ξ_0 *cannot be less than the sum of the least effective difference-weights* α_0, β_0, γ_0 *of the factor matrices A, B, C, D ; and every element of X whose difference-weight is less than this sum must be equal to 0.*

Many important conclusions can be drawn from this result. First let A, B, C, D be all simple pre-major slopes, so that

$$\alpha_0 \not< 0,\ \beta_0 \not< 0,\ \gamma_0 \not< 0,\ \delta_0 \not< 0.$$

Then if X is not a zero matrix, we must have $\xi_0 \not< 0$; and it follows that :

In a commutantal product of simple pre-major slopes, the product matrix is always a simple pre-major slope of the type appropriate to the product.

Next let A, B, C, D be all simple post-major slopes, so that

$$\alpha_0 \not< p-m,\ \beta_0 \not< q-p,\ \gamma_0 \not< r-q,\ \delta_0 \not< n-m.$$

Then if X is not a zero matrix, we must have $\xi_0 \not< n-m$; and it follows that :

In a commutantal product of simple post-major slopes, the product matrix is always a simple post-major slope of the type appropriate to the product.

Lastly let A, B, C, D be all simple (non-major) slopes, so that each of them is both a simple pre-major slope and a simple post-major slope. Then if X is not a zero matrix, we must have $\xi_0 \not< 0$, $\xi_0 \not< n-m$; and it follows that :

In a commutantal product of simple (non-major) slopes, the product matrix is always a simple (non-major) slope of the type appropriate to the product.

These three results show that :

All simple pre-major slopes, all simple post-major slopes, and all simple non-major slopes form three commutantal domains, the third domain being included in each of the other two.

If when the product (A) is particularised in any way, the parametric ranks of A, B, C, D, X are $\alpha, \beta, \gamma, \delta, \xi$, we have

$$\alpha_0 = p-\alpha,\ \beta_0 = q-\beta,\ \gamma_0 = r-\gamma,\ \delta_0 = n-\delta,\ \xi_0 = n-\xi,$$

and we can put

$$\alpha_0+\beta_0+\gamma_0+\delta_0 = n-\rho, \qquad \ldots \quad (3)$$

$$\rho = \alpha+\beta+\gamma+\delta-p-q-r. \qquad \ldots \quad (4)$$

When all possible values are given to $\alpha, \beta, \gamma, \delta$, the possible values of the integer ρ are those consistent with the conditions

$$\rho \not< -(p+q+r),\ \rho \not> m+p+q+r+n-4.$$

When X is not a zero matrix, we see from (3) and Theorem II that

we must have $\xi_0 \nless n-\rho$, and because we must also have $\xi_0 \ngtr n-1$, it follows that $\rho \nless 1$, or $\rho > 0$; *i.e.* we have the following theorem :

Theorem III. *If $\rho \ngtr 0$ in any particularisation of the product (A), then X is necessarily a zero matrix ; and whenever X is a non-zero matrix, we must have*

$$\rho > 0 ; \quad \text{and } \xi \nless 1, \quad \xi \ngtr \rho.$$

The integer ρ will occur in all the following articles. If we replace (A) by the commutantal product

$$\mathrm{I\ A\ B\ C\ D\ J = X}$$

in which $\mathrm{I} = [1]^{m}_{m}$ and $\mathrm{J} = [1]^{n}$, it is (see Art. 7) the discriminator of the product I A B C D J. It will be seen in the following articles that $\rho > 0$ is only a necessary and not a sufficient condition for X being a non-zero matrix.

5. The product matrix in a commutantal product of most general simple matrices having zero horizontal and vertical nullities and given parametric ranks.

We will next consider the commutantal product

$$[a]^{p}_{m}\ [b]^{q}_{p}\ [c]^{r}_{q}\ [d]^{n}_{r} = [x]^{n}_{m} \quad \text{or} \quad \mathrm{ABCD = X} \quad \ldots \quad (\mathrm{B})$$

in which A, B, C, D are four most general simple matrices having zero horizontal and vertical nullities and given parametric ranks α, β, γ, δ, so that the possible values of α, β, γ, δ are those consistent with the conditions

$$\alpha \nless m,\ \alpha \nless p,\ \alpha \ngtr m+p-1 ; \qquad \beta \nless p,\ \beta \nless q,\ \beta \ngtr p+q-1 :$$

$$\gamma \nless q,\ \gamma \nless r,\ \gamma \ngtr q+r-1 ; \qquad \delta \nless r,\ \delta \nless n,\ \delta \ngtr q+n-1 ; \quad \ldots \quad (1)$$

and as before we will put

$$\rho = \alpha+\beta+\gamma+\delta-p-r-q \quad \ldots \quad (2)$$

$$= m+(\alpha-m)+(\beta-p)+(\gamma-q)+(\delta-r) = (\alpha-p)+(\beta-q)+(\gamma-r)+(\delta-n)+n,$$

so that ρ is now a non-zero positive integer which satisfies the conditions

$$\rho \nless m,\ \rho \nless p,\ \rho \nless q,\ \rho \nless r,\ \rho \nless n,\ \rho \ngtr m+p+q+r+n-4.$$

The product (B) is derived from the product (A) by putting equal to 0 all elements (or parameters) of A, B, C, D having difference-weights less than $\alpha_0, \beta_0, \gamma_0, \delta_0$ respectively, where

$$\alpha_0 = p - a,\ \beta_0 = q - \beta,\ \gamma_0 = r - \gamma,\ \delta_0 = u - \delta, \quad \ldots \quad (3)$$

the other parameters of A, B, C, D remaining arbitrary and independent. It is assumed that A, B, C, D, X are expressed in the standard direct forms appropriate to their types, X having the type which renders the equation (B) commutantal.

When i and j are integers satisfying the conditions

$$i \not< 1,\ i \not> m,\ j \not< 1,\ j \not> n, \quad \ldots \quad (4)$$

the parameter x_{ij} of X is given by the formula (a) of Art. 4, and the expression (a) will contain a non-zero term $a_{ir}\, b_{uv}\, c_{ws}\, d_{vj}$ when and only when u, v, w, are integers satisfying the conditions

$$u \not< 1,\ u \not> p,\ ;\ v \not< 1,\ v \not> q\ ;\ w \not< 1,\ w \not> r\ ; \quad \ldots \quad (4')$$

$$u - i \not< p - a,\ v - u \not< q - \beta,\ w - v \not< r - \gamma,\ j - w \not< n - \delta \quad \ldots \quad (4'')$$

Again because the non-zero parameters of A, B, C, D are all arbitrary and independent, the expression (a) cannot vanish identically unless every one of its terms vanishes identically. Consequently x_{ij} will be a non-zero parameter of X when and only when there exist integers i, j, u, v, w, satisfying the conditions (4), (4'), (4''). Hence by eliminating u, v, w we see that when $\pm x_{ij}$ is a non-zero term of X the possible values of i and j, the possible values of the difference-weight

$$k = j - i, \quad \ldots \quad (5)$$

and the possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i \not< 1,\ k \not> m,\ j \not< 1,\ j \not> n,\ j - i \not< n - \rho\ ; \quad \ldots \quad (6)$$

$$k \not< 1 - m,\ i \not> m - \rho,\ k \not> n - 1\ ;\ \text{or } k \not< \xi_0,\ k \not> n - 1 \quad \ldots \quad (6')$$

$$i \not< 1,\ i \not< 1 - k,\ i \not> n - k,\ i \not> m\ ;\ j \not< 1,\ j \not< 1 + k,\ j \not> m + k,\ j \not> n \quad \ldots \quad (6'')$$

where ρ is the integer (2), and where ξ_0 is the greater of the two integers $1 - m$, $n - \rho$, and is the least difference-weight of a non-zero element of X. We could arrive at the same conclusions by observing as in Art. 4 that $j - i \not< n - \rho$ is a necessary condition that the expression (a) shall contain a non-zero term, and then using the formula (b). By eliminating first j and then i from the conditions

(6) we see that there is a non-zero element in every horizontal and every vertical row of X. In this way we obtain the following two theorems :

Theorem I. *In the commutantal product (B) the product matrix X is a non-zero simple matrix having zero horizontal and vertical nullities and parametric rank ξ, where*

$$\xi = n - \xi_0$$

is the smaller of the two non-zero positive integers

$$\rho,\ m+n-1.$$

the smallest difference-weight of a non-zero element of X being ξ_0, where

$$\xi_0 = n - \xi$$

is the greater of the two negative integers $n-\rho$, $1-m$.

Theorem II. *The elements of the ξ effective parametric diagonal lines of X are all different from 0, i.e. none of them vanish identically, and their parameters are given by the general formula*

$$x_{i,j} = \Sigma a_{i,u}\, b_{v,r}\, c_{r,w}\, d_{w,j}, \qquad (j-i=k) \qquad \ldots \quad (b)$$

where the possible values of i, j and k are given by (6), (6'), (6''), and where for a given pair of values of i and j the summation extends over all distinct sets of values of u, v, w which can be formed when u, v, w are elements of the respective sequences

$$[1, 2, \ldots p],\quad [1, 2, \ldots q],\quad [1, 2, \ldots r]$$

which make every one of the four terms on the left positive in the identical equation

$$\{(u-i)-\alpha_0\} + \{(v-u)-\beta_0\} + \{(w-v)-\gamma_0\} + \{(j-w)-\delta_0\} = k-(n-\rho). \quad \ldots \quad (b')$$

Here α_0, β_0, γ_0, δ_0 are the integers (3), which are the least difference-weights of non-zero elements of A, B, C, D.

Thus X has the form of a most general simple matrix having zero horizontal and vertical nullities and parametric rank ξ, in so far as none of its effective elements or parameters are equal to 0 ; but those effective parameters are rational integral functions of the effective parameters of A, B, C, D, and in saying that none of them are equal to 0 we mean that none of them vanish identically. When we particularise A, B, C, D by giving particular values to some of their effective parameters or by introducing relations between the effective parameters, some of the non-zero terms of X may be replaced by zero terms, but every zero element of X will necessarily remain a zero

element in the particularisation of X. Thus the particularisation of X may have non-zero horizontal and vertical nullities, and its parametric rank may be less than (but can never be greater than) the integer ξ defined in Theorem 1.

The number of terms in the expression on the left in (b) depends on the value of k. When i, j, k are given and satisfy the conditions (6), (6′), (6″), we can obtain all the terms by giving to u, v, w, in succession all integral values consistent with the mutually compatible conditions

$$\begin{aligned} &u \not< 1,\ u \not< i+a_0,\ u \not> p,\ u \not> j-\beta_0-\gamma_0-\delta_0\ ; \\ &v \not< 1,\ v \not< u+\beta_0,\ v \not> q,\ v \not> j-\gamma_0-\delta_0\ ; \\ &w \not< 1,\ w \not< v+\gamma_0,\ w \not> r,\ w \not> j-\gamma_0\ ; \end{aligned} \qquad \ldots \quad (1)$$

or to w, v, u, in succession all integral values consistent with the mutually compalible conditions

$$\begin{aligned} &w \not< 1,\ w \not< i+a_0+\beta_0+\gamma_0,\ w \not> r,\ w \not> j-\delta_0\ ; \\ &v \not< 1,\ v \not< i+a_0+\beta_0, \qquad v \not> q,\ v \not> w-\gamma_0\ ; \\ &u \not< 1,\ u \not< i+a_0, \qquad u \not> p,\ u \not> v-\beta_0. \end{aligned} \qquad \ldots \quad (1')$$

When $\rho \not> m+n-1$, *i.e.* when $\xi=\rho$ and $\xi_0=n-\rho$, the elements of the first effective parametric diagonal line of X, *i.e.* the elements of X having the least possible difference-weight ξ_0 of a non-zero element, are given by the formula

$$x_{ij}=\Sigma a_{ir}\, b_{ur}\, c_{rw}\, d_{wj}, \qquad (j-i=\xi_0=n-\rho), \qquad \ldots \quad (b_0)$$

where the possible values of i and j are those consistent with the conditions

$$i \not< 1-\xi_0,\ i \not> m\ ;\quad j \not< 1,\ j \not> m+\xi_0\ ;$$

and where $u=i+a_0=v-\beta_0,\ v=u+\beta_0=w-\gamma_0,\ w=v+\gamma_0=j-\delta_0,$

or

$$\begin{aligned} u &= i+a_0 &&= j-\beta_0-\gamma_0-\delta_0, \\ v &= i+a_0+\beta_0 &&= j-\gamma_0-\delta_0, \\ w &= i+a_0+\beta_0+\gamma_0 &&= j-\delta_0, \end{aligned}$$

there being only one term in the sum (b). In this case we have

$$\xi_0=a_0+\beta_0+\gamma_0+\delta_0.$$

Theorem I shows that :

The parametric rank ξ of X is that integer nearest in value to ρ which satisfies the necessary conditions

$$\xi \not< 0,\ \xi \not> m+n-1.$$

The least difference-weight ξ_0 of a non-zero element of X cannot be less then the sum $n-\rho$ of the least difference-weights of non-zero elements of A, B, C, D, and is that integer nearest in value to $n-\rho$ which satisfies the necessary conditions

$$\xi_0 \not< 1-m,\ \xi_0 \not> n-1.$$

When A, B, C, D are particularised, their nullities remaining unaltered, these statements give respectively the greatest possible value of ξ and the least possible value of ξ_0.

6. Some auxiliary theorems.

The lemmas to be now given are used in obtaining the result of the next article. In each of them it is to be understood that $A=[a]^{\alpha}_{m}$ is a most general direct simple matrix of any non-ruled type θ having zero horizontal and vertical nullities and given parametric rank α, so that α is a positive integer whose possible values are those consistent with the conditions:

$$\alpha \not< m,\ \alpha \not< n,\ \alpha \not> m+n-1.$$

The lemmas remain true when θ is any ruled type provided that we make no distinction between the two types $\pm\theta$, *i. e.* between two types such as $\{\pi, -\pi'\}$, $\{-\pi, \pi'\}$ each of which is derived from the other by changing the signs of both its elements. It is of course to be understood as always that $m \not< 1$ and $n \not< 1$.

Lemma A. *Let $(A')=[a']^{\alpha}_{m-x}$ be the matrix formed by striking out the first x horizontal parametric rows of A, where*

$$x \not< 0,\ x \not> m-1\ ;\quad \textit{and therefore}\quad x \not> \alpha-1.$$

Then A' is a most general non-zero simple matrix of type θ having parametric rank $\alpha-x$, horizontal nullity 0, vertical nullity k,

where k is the greater of the two integers 0 and $x-(\alpha-n)$,

so that $k=0$ when $x \not> \alpha-n$, $k=x-(\alpha-n)$ when $x \not< \alpha-n$.

It is sufficient to observe that each time an initial horizontal parametric row is struck out, the parametric rank is reduced by 1 ; and that the number of initial horizontal parametric rows in A which contain no pre-parametric zero elements is $\alpha+1-n$. In the excluded case when $x=n$, the matrix A′ is non-existent.

When A is of the ordinary type $\{\pi, \pi\}$, we can represent it and A′ by hemipteric class-symbols of the first type in the forms

$$A=\binom{n}{m}=\binom{k,\ n-k}{x,\ m-x}, \qquad A'=\binom{k,\ n-k}{m-x},$$

where the effective constituent $A_1=\binom{n-k}{m-x}$ of A′ is a most general simple marix having zero horizontal and vertical nullities and parametric rank $\alpha-x$, A_1 being a simple pre-major slope when $k\not<0$, *i.e.* when $x\not<\alpha-n$. If we put

$$A=\begin{bmatrix} n, & \times \\ 0, & a' \end{bmatrix}^{k,\ n-k}_{u,\ m-x}, \quad A'=[0,\ a']^{k,\ n-k}_{m-x}, \quad A_1=[a']^{n-k}_{m-x},$$

the first diagonal constituent U of A (when it exists, *i. e.* when $k>0$, or $x>\alpha-n$) is a general simple post-major slope of type θ, having parametric rank $u=\alpha-(n-k)$ and zero horizontal and vertical nullities. When A is a general simple pre-major or post-major or non-major slope, the same is true of A_1.

Lemma B. *Let* $A''=[a'']^{n-y}_{m}$ *be the matrix formed by striking out the last* y *vertical parametric rows of* A, *where*

$$y\not<0,\ y\not>n-1 ; \qquad \text{so that} \quad y\not>\alpha-1.$$

Then A'' *is a most general non-zero simple matrix of type* θ *having parametric rank* $\alpha-y$, *horizontal nullity* h, *vertical nullity* 0, *where* h *is the greater of the two integers* 0 *and* $y-(\alpha-m)$, *so that* $h=0$ *when* $y\not>\alpha-m$, $h=y-(\alpha-m)$ *when* $y\not<\alpha-m$.

It is sufficient to observe that each time a final vertical parametric row is struck out, the rank is reduced by 1 ; and that the number of final vertical parametric rows in A which contain no post-parametric zero elements is $\alpha+1-m$. In the excluded case when $y=n$, A″ is non-existent.

When A is of the ordinary type $\{\pi, \pi\}$, we can represent it and A″ by hemipteric class-symbols of the first type in the forms

$$A=\binom{n}{m}=\binom{n-y,\ y}{m-h,\ h}\ ,\qquad A''=\binom{n-y}{m-h,\ h}\ ,$$

where the effective constituent $A_2=\binom{n-y}{m-h}$ of A″ is a most general simple matrix having zero horizontal and vertical nullities and parametric rank $a-y$, A_2 being a general simple post-major slope when $h\not<0$, *i.e.* when $y\not<a-m$. If we put

$$A=\begin{bmatrix} a'', & \times \\ 0, & \eta \end{bmatrix}^{n-y,\ y}_{m-h,\ h},\quad A''=\begin{bmatrix} a'' \\ 0 \end{bmatrix}^{n-y}_{m-h,\ h},\quad A_2=[a'']^{n-y}_{m-h},$$

the last diagonal constituent V of A (when it exists, *i.e.* when $h>0$, or $y>a-m$) is a general simple pre-major slope of type θ, having parametric rank $y=a-(m-h)$ and zero horizontal and vertical nullities. When A is a general simple pre-major or post-major or non-major slope, the same is true of A_2.

Lemma C. *Let* $C=[c]^{n-y}_{m-y}$ *be the matrix formed by striking out the first* x *horizontal and last* y *vertical parametric rows of* A, *where* $x\not<0$, $x\not>m-1$; $y\not<0$, $y\not>n-1$; $x+y\not>a-1$; *so that* $x+y\not>m+n-1$. *Then C is a most general non-zero simple matrix of type* θ *having*

parametric rank $a-x-y$, *horizontal nullity* h, *vertical nullity* k,

where h *is the greater of the two integers* 0 *and* $y-(a-m)$,

and k *is the greater of the two integers* 0 *and* $x-(a-n)$.

This can be proved by successive applications of Lemmas A and B. In the excluded cases when $x=m$ or $y=n$, C is non-existent. In the excluded case when $x+y>a-1$, C is a zero matrix.

When A is of the ordinary type $\{\pi, \pi\}$, we can represent A and C by hemipteric class-symbols of the first type in the forms

$$A=\binom{n}{m}=\binom{k,\ n-k-y,\ y}{x,\ m-h-x,\ h},\quad C=\binom{k,\ n-k-y}{m-h-x,\ h},$$

where the effective constituent $C_2=\binom{n-k-y}{m-h-a}$ of C is a most general simple matrix having zero horizontol and vertical nullities and

parametric rank $\alpha-x-y$, C_0 being a general simple pre-major slope when $x \not< \alpha-n$ or $k=x-(\alpha-n)$, a general simple post-major slope when $y \not< \alpha-m$ or $h=y-(\alpha-m)$, and a general simple non-major slope when both $x \not< \alpha-n$ and $y \not< \alpha-m$. If we put

$$A=\begin{matrix} a, \times, \times \\ 0, c, \times \\ 0, 0, v \end{matrix}\Bigg|\begin{matrix} k, n-k-y, y \\ \\ r, m-x-h, h \end{matrix} \qquad C=\begin{bmatrix} 0, & c \\ 0, & 0 \end{bmatrix}\begin{matrix} k, n-k-y \\ m-x-h, h \end{matrix}$$

$$C_0=[c]^{n-k-y}_{m-x-h},$$

the first diagonal constituent U of A (when it exists, *i.e.* when $k>0$) is a general simple post-major slope of type θ, and the last diagonal constituent V (when it exists, *i.e.* when $h>0$) is a general simple pre-major slope of type θ. When A is a general simple pre-major or post-major or non-major slepe, the same is true of C_0.

7. The product matrix in a commutantal product of most general simple matrices having given horizontal and vertical nullities and given parametric ranks.

In each of the factor matrices all three of the nullities (basical, horizontal, vertical) are given, but instead of regarding the basical nullities as given, it is more convenient to regard the parametric (or apical) ranks as given. Accordingly we will now consider the commutantal product

$$[a]^{p}_{m}\ [b]^{q}_{p}\ [c]^{r}_{q}\ [d]^{n}_{r}=[x]^{n}_{m} \quad \text{or} \quad ABCD=X \qquad \ldots \quad (C)$$

in which A, B, C, D are four most general simple matrices expressed in the standard direct forms appropriate to their types and having given parametric ranks

$$\alpha, \beta, \gamma, \delta, \qquad \ldots \quad (\alpha_0)$$

given horizontal nullities $\alpha_1, \beta_1, \gamma_1, \delta_1$, and given vertical nullities $\alpha_2, \beta_2, \gamma_2, \delta_2$. The post-major and pre-major parametric excesses of their effective constituents are the essentially positive integers

$$\alpha_{10}=\alpha+\alpha_1-m,\ \beta_{10}=\beta+\beta_1-p,\ \gamma_{10}=\gamma+\gamma_1-q,\ \delta_{10}=\delta+\delta_1-r \ldots \quad (1)$$

$$\alpha_{20}=\alpha+\alpha_2-p,\ \beta_{20}=\beta+\beta_2-q,\ \gamma_{20}=\gamma+\gamma_2-r,\ \delta_{20}=\delta+\delta_2-n\ ; \ldots \quad (1')$$

their basical nullities are the essentially positive integers, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$, given by

$$\alpha+\bar{\alpha}=m+p-1,\ \beta+\bar{\beta}=p+q-1,\ \gamma+\bar{\gamma}=q+r-1,\ \delta+\bar{\delta}=r+n-1,$$

and their least effective difference-weights are the integers

$$\alpha_0=p-\alpha,\ \beta_0=q-\beta,\ \gamma_0=r-\gamma,\ \delta_0=n-\delta \quad \ldots \quad (1'')$$

each of which may be either positive or negative. The product (C) is derived from the product (A) of Art. 4 by putting all elements of A, B, C, D lying in their first α, β, γ, δ, parametric diagonal lines (counting from the base), their last α_1, β_1, γ_1, δ_1, horizontal parametric rows, and their first α_2, β_2, γ_2, δ_2 vertical parametric rows equal to 0, the parameters of their other elements (whose difference-weights cannot be less than α_0, β_0, γ_0, δ_0 respectively) remaining arbitrary and independent. The product matrix X is to be regarded as having the type which renders the equation (C) commutantal, and is supposed to be expressed in the direct standard form appropriate to that type. The parametric rank, horizontal nullity and vertical nullity of X (which have to be determined) will be denoted by ξ, ξ_1 and ξ_2, these being essentially positive integers. Then the post-major and pre-major parametric excesses of the least effective constituent of X will be the essentially positive integers

$$\xi_{10}=\xi+\xi_1-m,\ \xi_{20}=\xi+\xi_2-n \quad \ldots \quad (2)$$

and the basical nullity and least effective difference-weight of X will be the integers $\bar{\xi}$ and ξ_0 given by

$$\xi+\bar{\xi}=m+n-1,\quad \xi_0=n-\xi,$$

$\bar{\xi}$ being essentially positive, and ξ_0 being subject to the conditions

$$\xi_0 \not< 1-m,\quad \xi_0 \not> n.$$

If A is a zero matrix, then X is also a zero matrix, and we have

$$\alpha=0,\ \alpha_1=m,\ \alpha_2=p,\ \alpha_{10}=0,\ \alpha_{20}=0,\ \alpha_0=p,\ \alpha-\alpha_{10}-\alpha_{20}=0. \quad \ldots \quad (3)$$

If A is not a zero matrix, the possible values of α, α_1, α_2 are those consistent with the conditions

$$\alpha_1 \not< 0,\ \alpha_2 \not< 0,\ \alpha+\alpha_1 \not< m,\ \alpha+\alpha_2 \not< p,\ \alpha+\alpha_1+\alpha_2 \not> m+p-1\ ; \quad \ldots \quad (4)$$

which include $\alpha \not< 1$, $\alpha \not> m+p-1$; $\alpha_1 \not< 0$, $\alpha_2 \not> m-1$, $\alpha_2 \not< 0$, $\alpha_2 \not> p-1$; the possible values of α, α_{10}, α_{20} are those consistent with the conditions

$$\alpha_{10} \not< 0,\ \alpha_{10} \not< \alpha-m,\ \alpha_{20} \not< 0,\ \alpha_{20} \not< \alpha-p,\ \alpha_{10}+\alpha_{20} \not> \alpha-1\ ; \quad \ldots \quad (4')$$

which include

$$\alpha \nless 1,\ \alpha \ngtr m+p-1;\ \alpha_{10} \nless 0,\ \alpha_{10} \ngtr p-1,\ \alpha_{20} \nless 0,\ \alpha_{20} \ngtr m-1;$$

and it is to be observed in particular that we have

$$\alpha \nless 1,\ \alpha-\alpha_{10}+\alpha_{20} \ngtr 1. \qquad \ldots \quad (4'')$$

We shall usually regard α_1, α_2, α, α_n as being expressed in terms of α, α_{10} α_{20}.

We have corresponding results for B, C, D, which show that two corresponding integers in the set (α_n) and the set

$$\alpha-\alpha_{20}-\alpha_{10},\ \beta-\beta_{20}-\beta_{10},\ \gamma-\gamma_{20}-\gamma_{10},\ \delta-\delta_{20}-\delta_{10} \qquad \ldots \quad (a_1)$$

are either both equal to 0 or both greater than 0; and that when A, B, C, D are all non-zero matrices, all the integers (a_0) and (a_1) are greater than 0. The product matrix only needs evaluation when A, B, C, D are non-zero matrices, so that the conditions (4), (4') (4'') and the corresponding conditions for B, C, D are all satisfied; and this is the case when and only when all the integers (a_n) and (a_1) are greater than 0.

If for the sake of brevity we put

$$\rho_{\alpha\beta} = \alpha+\beta-\mu,\ \rho_{\beta\gamma} = \beta+\gamma-q,\ \rho_{\gamma\delta} = \gamma+\delta-r,$$

$$\rho_{\alpha\beta\gamma} = \alpha+\beta+\gamma-q,\ \rho_{\beta\gamma\delta} = \beta+\gamma+\delta-r,$$

$$\rho = \rho_{\alpha\beta\gamma\delta} = \alpha+\beta+\gamma+\delta-\mu-q-r, \qquad \ldots \quad (5)$$

it will be found that the general character of X depends on the values of the integers

$$\rho_{\alpha\beta}-\alpha_{20}-\beta_{10},\ \rho_{\beta\gamma}-\beta_{20}-\gamma_{10},\ \rho_{\gamma\delta}-\gamma_{20}-\delta_{10}, \qquad \ldots \quad (a_2)$$

$$\rho_{\alpha\beta\gamma}-\alpha_{20}-\gamma_{10},\ \rho_{\beta\gamma\delta}-\beta_{20}-\delta_{10}, \qquad \ldots \quad (a_3)$$

$$\rho_{\alpha\beta\gamma\delta}-\alpha_{20}-\delta_{10} = \rho-\alpha_{20}-\delta_{10}, \qquad \ldots \quad (a_4)$$

which will be called the *discriminators* of the partial products AB, BC, CD, ABC, BCD, ABCD formed with consecutive factors of the product (C), and the values of the integers

$$\alpha-\alpha_{10},\ \rho_{\alpha\beta}-\beta_{10},\ \rho_{\alpha\beta\gamma}-\gamma_{10},\ \rho_{\alpha\beta\gamma\delta}-\delta_{10}, \qquad \ldots \quad (b_1)$$

$$\delta-\delta_{20},\ \rho_{\gamma\delta}-\delta_{20},\ \rho_{\beta\gamma\delta}-\beta_{20},\ \rho_{\alpha\beta\gamma\delta}-\alpha_{20}, \qquad \ldots \quad (b_2)$$

which, if $\mathrm{I}=[1]$ and $\mathrm{J}=[1]$,

so that $$X=IABCDJ,$$

are the discriminators of the partial products

IA, IAB, IABC, IABCD, and DJ, CDJ, BCDJ, ABCDJ.

The integer ρ is the discriminator of the product IABCDJ.

In fact if, when A, B, C, D, are all non-zero matrices, we first: cancel in succession the (passive) initial vertical zero major-parametric rows of A, B, C with the corresponding (passive) initial horizontal major-parametric rows of B, C, D, applying Lemma A of Art. 6 to the effective constituents of B, C, D when they are not thereby annihilated, and stopping as soon as the annihilation of one of them shows X to be a zero matrix;

and, when X has not been thus shown to be a zero matrix, next: cancel in succession the (passive) final horizontal zero major-parametric rows of D, C, B with the corresponding (passive) final vertical major-parametric rows of C, B, A, applying Lemma B to the effective constituents of C, B, A, which will never be annihilated;

we shall obtain the following theorem:

Theorem I. *Let θ be the symbolic commutant which is the product of the symbolic commutants representing the types of the successive factor matrices A, B, C, D in the commutantal product* (C). *Then in evaluating the product matrix X we can distinguish between two cases:*

CASE I. *If any one of the discriminators (a_2), (a_3), (a_4), or if we please any one of the discriminators (a_1), (a_2), (a_3), (a_4), is not greater than 0, X must be a zero matrix.*

CASE II. *If every one of the discriminators (a_2), (a_3), (a_4), or if we please every one of the discriminators (a_1), (a_2), (a_3), (a_4), is greater than 0, then X is non-zero simple matrix of the type θ having horizontal nullity ξ_1, vertical nullity ξ_2, and parametric rank ξ, where:*

(1) *$m-\xi_1$ is the smallest of the non-zero poitive integers (b_1) (the condition $m-\xi_1 \not> \alpha-\alpha_{10} \not> m-\alpha_1$ showing that ξ_1 is positive);*

(2) *$n-\xi_2$ is the smallest of the non-zero positive integers (b_2), (the condition $n-\xi_2 \not> \delta-\delta_{10} \not> n-\delta_2$ showing that ξ_2 is positive);*

(3) *ξ is the smaller of the two non-zero positive integers*

$$\rho=\alpha+\beta+\gamma+\delta-v-q-r, \quad \sigma=(m-\xi_1)+(n-\xi_2)-1;$$

and none of the elements of the ξ effective parametric diagonal lines of

the least effective constituent X_0 of X vanish identically, i.e. X has the form of a most general simple matrix having horizontal nullity ξ_1, vertical nullity ξ_2, and parametric rank ξ.

NOTE—*Alternative form of Theorem I.*

Let ω be the smallest of the discriminators (a_1), (a_2), (a_3), (a_4); let ω_1 be the smallest of the discriminators (b_1); let ω_2 be the smallest of the discriminators (b_2); and let ρ be the discriminator of the product IABCDJ; so that

$$\rho \not< \omega_1 \not< \omega, \quad \rho \not< \omega_2 \not< \omega.$$

Then if $\omega \not> 0$, X is zero matrix; and if $\omega > 0$, X is a non-zero simple matrix of type θ having horizontal nullity $\xi_1 = n - \omega_1$, vertical nullity $\xi_2 = n - \omega_2$, and parametric rank ξ, where ξ is that integer nearest in value to ρ which (see Ex. iii. of Art. **3**) satisfies the necessary condition

$$\xi \not> \omega_1 + \omega_2 - 1.$$

We will consider first the trivial case in which one of the factor matrices A, B, C, D is a zero matrix, *i.e.* one of the discriminators (a) is not greater than 0, so that X is necessarily a zero matrix. Let P and Q be any two consecutive factor matrices. Then if either P or Q is a zero matrix, *i.e.* if the discriminator of either P or Q is not greater than 0, the discriminator of the product PQ cannot be greater than 0. For example if B is a zero matrix, we have $\beta = 0$, $\beta_{10} = 0$, $\beta_{20} = 0$, and therefore

$$\rho_{\alpha\beta} - \alpha_{20} - \beta_{10} = (\alpha + \beta - p) - \alpha_{20} - \beta_{10} = \alpha - \alpha_{20} - p = -\alpha_2 \not> 0,$$

$$\rho_{\beta\gamma} - \beta_{20} - \gamma_{10} = (\beta + \gamma - q) - \beta_{20} - \gamma_{10} = \gamma - \gamma_{10} - q = -\gamma_1 \not> 0,$$

i.e. the discriminators of the products AB, BC, cannot be greater than 0. These considerations show that Theorem I is true when A, B, C, D are not all zero matrices, Case I only being possible. They show further that every one of the discriminators (a_2), (a_3), (a_4) is greater than 0 when and only when every one of the discriminators (a_1), (a_2), (a_3), (a_4) is greater than 0, and also when and only when every one of the integers (a_0), (a_1), (a_2), (a_3), (a_4) is greater than 0. Hence in proving the theorem we may suppose the discriminators mentioned in it to be (a_2), (a_3), (a_4); but it makes no difference if we add to them the discriminators (a_1) or the discriminators (a_1) and the integers (a_0).

Again if the theorem is true for the correlated commutantal product $A'B'C'D' = X'$ in which A', B', C', D' are all of the ordinary type $\{\pi, \pi\}$, it will be true for the given commutantal product (C). For then X in the product (C) will be the simple matrix of type θ correlated with the simple matrix X' of type $\{\pi, \pi\}$, and X will have

the same orders, the same rank, same horizontal and vertical nullities, and the same parametric rank as X′. Moreover the expression for any parameter x_{ij} of X as a rational integral function of the parameters of A, B, C, D will be identical with the expression for the parameter x_{ij} of X′ as a rational integral function of the parameters of A′, B′, C′, D′.

Thus in proving Theorem I and the following theorems of this article, we may suppose that A, B, C, D are all non-zero matrices, so that all the conditions such as (4), (4′), (4″), are satisfied ; and we may further suppose that A, B, C, D are all of the ordinary type $\{\pi, \pi\}$.

When those suppositions are made, we have to determine X from an equation of the form

$$X=ABCD=\begin{bmatrix} 0, & a' \\ 0, & 0 \end{bmatrix}^{\alpha_2,\ p-\alpha_2}_{m-\alpha_1,\ \alpha_1} \quad \begin{bmatrix} 0, & b' \\ 0, & 0 \end{bmatrix}^{\beta_2,\ q-\beta_2}_{p-\beta_1,\ \beta_1}$$

$$\begin{bmatrix} 0, & c' \\ 0, & 0 \end{bmatrix}^{\gamma_2,\ r-\gamma_2}_{q-\gamma_1,\ \gamma_1} \quad \begin{bmatrix} 0, & d' \\ 0, & 0 \end{bmatrix}^{\delta_2,\ n-\delta_2}_{r-\delta_1,\ \delta_1}$$

or, when we represent A, B, C, D by hemipteric class-symbols of the first type, from an equation of the form

$$X=ABCD=\begin{pmatrix} \alpha_2, & p-\alpha_2 \\ m-\alpha_1, & \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_2, & q-\beta_2 \\ p-\beta_1, & \beta_1 \end{pmatrix} \begin{pmatrix} \gamma_2, & r-\gamma_2 \\ q-\gamma_1, & \gamma_1 \end{pmatrix}$$

$$\begin{pmatrix} \delta_2, & n-\delta_2 \\ r-\delta_1, & \delta_1 \end{pmatrix},$$

in which the effective constituents of A, B, C, D are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and the given parametric ranks

$$\alpha, \quad \beta, \quad \gamma, \quad \delta.$$

The successive cancellations mentioned above can now be easily carried out by utilising the properties of hemipteric class-symbols of the first type, the cancellations of corresponding passive rows being represented by cancellations of the corresponding index numbers.

When it is necessary to carry through all the cancellations of the first set, we cancel in succession the α_1, P_2, Q_2 initial vertical rows of 0's in A, B, C with the corresponding initial horizontal rows of

B, C, D, thus reducing the equation determining X to the form

$$X=A'B''C''D'=\begin{pmatrix}p-a_2\\ m-a_1,\ a_1\end{pmatrix}\begin{pmatrix}q-P_2\\ p-a_2-\beta_1,\ \beta_1\end{pmatrix}\begin{pmatrix}r-Q_2\\ q-P_2-\gamma_1,\ \gamma_1\end{pmatrix}$$

$$\begin{pmatrix}R_2,\ n-k\\ r-Q_2-\delta_1,\ \delta_1\end{pmatrix},$$

where $P_2=p_2+\beta_2$ is the greater of the two integers $\beta_2,\ a_2+(q-\beta)$,

$Q_2=q_2+\gamma_2$ is the greater of the two integers $\gamma_2,\ P_2+(r-\gamma)$,

$R_2=r_2+\delta_2$ is the greater of the two integers $\delta_2,\ Q_2+(n-\delta)$,

and where the integers $p_2,\ q_2,\ r_2$ thus defined are the numbers of the additional initial vertical rows of 0's which the successive cancellations introduce into B, C, D. The successive cancellations show that X is necessarily a zero matrix unless the successive conditions

$$p-a_2+\beta_1>0,\ q-P_2-\gamma_1>0,\ r-Q_2-\delta_1>0$$

are all satisfied ; and since (using the definitions of P_2 and Q_2)

$p-a_2-\beta_1$ is the discriminator of AB

$q-P_2-\gamma_1$ is the smaller of the discriminators of BC, ABC,

$r-Q_2-\delta_1$ is the smallest of the discriminators of CD, BCD, ABCD,

it follows that Theorem I is true as regards Case I. When all these conditions are satisfied, the effective constituents of A', B'', C'', D' are most general non-zero simple matrices of the type $\{\pi,\ \pi\}$ having zero horizontal and vertical nullities and parametric ranks

$$\alpha,\quad \beta-a_2,\quad \gamma-P_2,\quad \delta-Q_2.$$

When the discriminators (a_2), (a_2), (a_1) are all greater than 0, and all the preceding cancellations have been carried through, we proceed to the second set of cancellations. We cancel in succession the δ_1, R_1, Q_1 final horizontal rows of 0's in D', C'', B'' with the corresponding final vertical rows of C'', B'', A', thus reducing the equation determining X to the form

$$X=A_1\ B_0\ C_0\ D_0$$

$$=\begin{pmatrix}p-a_2-Q_1\\ m-P_1,\ P_1\end{pmatrix}\begin{pmatrix}q-P_2-R_1\\ p-a_1-Q_1\end{pmatrix}\begin{pmatrix}r-Q_2-\delta_1\\ q-P_2-R_1\end{pmatrix}\begin{pmatrix}R_2,\ n-R_2\\ r-Q_2-\delta_1\end{pmatrix}$$

where $R_1 = r_1 - \gamma_1$ is the greater of the two integers

$$\gamma_1, (q-\gamma)+\delta_1,$$

$Q_1 = q_1 + \beta_1$ is the greater of the two integers

$$\beta_1, (p-\beta)+R_1,$$

$P_1 = p_1 + \alpha_1$ is the greater of the two integers

$$\alpha_1, (m-\alpha)+Q_1,$$

and where the integers r_1, q_1, p_1 thus defined are the number of the additional final horizontal rows of 0's which the successive cancellations introduce into C″, B″, A′. The successive cancellations show that X is necessarily a zero matrix unless the successive conditions

$$r-Q_2-\delta_1>0, \quad q-P_2-R>0, \quad p-\alpha_2-Q_1>0$$

are all satisfied ; but since

$r-Q_2-\delta_1$ is the smallest of the discriminators of

CD, BCD, ABCD,

$q-P_2-R_1$ is the smallest of the discriminators of

BC, ABC, BCD, ABCD,

$p-\alpha_2-Q_1$ is the smallest of the discriminators of

AB, ABC, ABCD,

these conditions are all satisfied by supposition. After these cancellation have been completed, the effective constituents A_0, B_0, C_0, D_0 of A_1, B_0, C_0, D_2 are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and parametric ranks

$$\alpha-Q_1, \quad \beta-\alpha_2-R_1, \quad \gamma-P_2-\delta_1, \quad \delta-Q_2.$$

Thus if we put $\xi_1 = P_1$, $\xi_2 = R_2$,

and $X = ABCD = IABCDJ$, where $I = [1]_m^m$ and $J = [1]_n^n$,

we see that in Case II of Theorem I the product matrix X is determined by the equation

$$X = A_1 B_0 C_0 D_2 = \binom{p_0}{m-\xi_1, \xi_1} \binom{q_0}{p_0} \binom{r_0}{q_0} \binom{\xi_2, n-\xi_2}{r_0}$$

$$= \binom{p_0}{m_0, P_1} \binom{q_0}{p_0} \binom{r_0}{q_0} \binom{R_2, n_0}{r_0}, \quad \ldots \quad (D)$$

where

$m_0 = m - \xi_1 = m - P_1$ is the smallest of the integers (δ_1) which are the discriminators of the partial products

IA, IAB, IABC, IABCD containing IA,

$p_0 = p - \alpha_2 - Q_1$ is the smallest of the discriminators of the partial products

AB, ABC, ABCD containing AB,

$q_0 = q - P_2 - R_1$ is the smallest of the discriminators of the partial products

BC, ABC, BCD, ABCD containing BC,

$r_0 = r - Q_2 - \delta_1$ is the smallest of the discriminators of the partial products

CD, BCD, ABCD, containing CD,

$n_0 = n - \xi_2 = n - R_2$ is the smallest of the integers (δ_1) which are the discriminators of the partial products

DJ, CDJ, BCDJ, ABCDJ containing DJ ;

all these discriminators being non-zero positive integers ; and where

$$A_0 = \begin{pmatrix} p_0 \\ m - \xi_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ m_0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} r_0 \\ q_0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} n_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} n - \xi_2 \\ r_0 \end{pmatrix}$$

are most general non-zero simple matrices of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities and parametric ranks

$$\alpha - Q_1, \quad \beta - \alpha_2 - R_1, \quad \gamma - P_2 - \delta_1, \quad \delta - Q_2.$$

The integers ξ_1 and ξ_2 are those defined in Theorem I.

Since X as given by the formula (D) has ξ_1 final horizontal and ξ_2 initial vertical rows of 0's we can write

$$X = \begin{bmatrix} 0, & x'' \\ 0, & 0 \end{bmatrix}_{m-\xi_1,\ \xi_1}^{\xi_2,\ n-\xi_2} = \begin{pmatrix} \xi_2, & n-\xi_2 \\ m-\xi_1, & \xi_1 \end{pmatrix},$$

$$X_0 = [x'']_{m-\xi_1}^{n-\xi_2} = \begin{pmatrix} n-\xi_2 \\ m-\xi_1 \end{pmatrix},$$

and replace (D_1) by the equation

$$\begin{pmatrix} \xi_2, & n-\xi_2 \\ m-\xi_1, & \xi_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ m-\xi_1, & \xi_1 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \begin{pmatrix} r_0 \\ q_0 \end{pmatrix} \begin{pmatrix} \xi_2, & n-\xi_2 \\ & r_0 \end{pmatrix}$$

or

$$\begin{bmatrix} 0, & x'' \\ 0, & 0 \end{bmatrix}^{\xi_2,\ n-\xi_2}_{m-\xi_1,\ \xi_1} = \begin{bmatrix} a'' \\ 0 \end{bmatrix}^{p_0}_{m-\xi_1,\ \xi_1} [b'']^{q_0}_{p_0} [c'']^{r_0}_{q_0} [0, d'']^{\xi_2,\ n-\xi_2}_{r_0} \qquad (D')$$

and when we cancel corresponding active rows of θ's on both sides, we see that X_0 is the product matrix in the commutantal product

$$[a'']^{p_0}_{m-\xi_1} [b'']^{q_0}_{p_0} [c'']^{r_0}_{q_0} [d'']_{r_0} = [x'']_{m-\xi_1}$$

or $$A_0B_0C_0D_0 = X . \qquad \ldots \quad (C_0)$$

which is one of the kind considered in Art. 5. Observing that

$$\{(\alpha-Q_1)+(\beta-\alpha_2-R_1)+(\gamma-P_2-\delta_1)+\delta-(Q_2)\}-(p_0+q_0+r_0)=\rho,$$

this being a necessary consequence of the characters of the passive cancellations, it follows from Art. 5 that X_0 is a non-zero simple matrix of the type $\{\pi, \pi\}$ having zero horizontal and vertical nullities, and parametric rank ξ, where ξ is the smaller of the two integers

$$\rho, \sigma = (m-\xi_1)+(n-\xi_2)-1,$$

and that none of the elements forming the ξ effective parametric diagonal lines of X_0 are equal to 0. Thus Theorem I is also true as regards Case II.

In proving Theorem 1 we have also proved the following second theorem :

Theorem II. *In Case II of Theorem 1, i.e. when the discriminators of all the products AB, BC, CD, ABC, BCD, ABCD are greater than 0, the effective constituent X_0 of the product matrix X in the commutantal product* (C) *is the product matrix in the commutantal product*

$$[a'']^{p_0}_{m_0} [b'']^{q_0}_{p_0} [c'']^{r_0}_{q_0} [d'']^{n_0}_{r_0} = [x'']^{n_0}_{m_0} \qquad \text{or} \qquad A_0B_0C_0D_0 = X_0,$$

where A_0, B_0, C_0, D_0 *are the general non-zero simple matrices with zero horizontal and vertical nullities and with parametric ranks*

$$\alpha-Q_1, \quad \beta-\alpha_2-R_1, \quad \gamma-P_2-\delta_1, \quad \delta-Q_2$$

which are formed from A, B, C, D *by striking out from each of them as many initial vertical, initial horizontal, final vertical and final horizontal major-parametric rows as are indicated in the following table where the last column shows the number of rows of each kind which must be struck out from* X *to form* X_0 :

		A	B	C	D	X
Initial rows	*vertical*	$M_2=\alpha_2$	P_2	Q_2	$R_2=\xi_2$	ξ_2
	horizontal	——	$M_2=\alpha_2$	P_2	Q_2	—
Final rows	*vertical*	Q_1	R_1	$N_1=\delta_1$	——	—
	horizontal	$P=\xi_1$	Q_1	R_1	$N_1=\delta_1$	ξ_1

If we suppose A_0, B_0, C_0, D_0, X_0 to be expressed in the standard direct forms appropriate to their types, and denote their least effective difference-weights under these circumstance by α_0'', β_0'', γ_0'', δ_0'', ξ_0'' so that

ξ_0'' is the greater of the two integers $\quad n_0-\rho, \quad 1-m_0,$

$$\alpha_0''=p_0-\alpha-Q_1=\alpha_0-\alpha_2, \quad \beta_0''=q_0-\beta+\alpha_2+R_1=\beta_0+\alpha_2-P_2,$$

$$\gamma_0''=r_0-\gamma+P_2+\delta_1=\gamma_0+P_2-Q_2, \quad \delta_0''=n_0-\delta+Q_2=\delta_0+Q_2-R_2,$$

$$\xi_0''=n_0-\xi=\xi_0-\xi_2,$$

we know from Art. 5 that if x''_{IJ} is an element of one of the ξ effective parametric diagonal lines of X_0, *i.e.* a non-zero element of X_0, the possible values of I and J, the possible values of the difference-weight $k''=J-I$, and the possible values of I and J when k'' is given are those consistent with the respective sets of conditions

$$I\not<1, \ I\not>m_0, \ J\not<1, \ J\not>n_0, \ J-I\not<n_0-\rho\ ;$$

$$k''\not<\xi''_0, \ k''\not>n_0-1\ ;$$

$$I\not<1, \ I\not<1-k'', \ I\not>n_0-k'', \ I\not>m_0\ ; \ J=I+k'',$$

and we know further that when I, J, k'' have any given values consistent with these conditions, the expression for the parameter x''_{IJ} is

$$x''_{IJ}=\Sigma a''_{IU}\ b''_{VU}\ c''_{VW}\ d''_{WJ}\ , \qquad (J-I=k''),$$

where the summation extends over all distinct sets of values of U, V W which can be formed which U, V, W are elements of the respective sequences

$$[1, 2,\ldots p_0], \quad [1, 2,\ldots q_0], \quad [1, 2,\ldots r_0]$$

which make every term on the left positive in the equation

$$\{(U-I)-\alpha_0''\}+\{(V-U)-\beta_0''\}+\{(W-V)-\gamma_0''\}+\{(J-W)-\delta_0''\}$$
$$=k''-(n_0-\rho$$

Now let A, B, C, D, X be expressed in the standard direct forms appropriate to their types as in the previous theorems. Then

$$a''_{IU}=a_{in}, \ b''_{UV}=b_{uv}, \ c''_{VW}=c_{vw}, \ d''_{WJ}=d_{wj},$$

$$x''_{IJ}=x_{ij},$$

where

$$i=I, \ u=\alpha_0+U, \ v=P_2+V, \ w=Q_2+W, \ j=R_2+J=\xi_2+J\ ;$$

and if $\qquad k=j-i, \quad k''=J-I$

are the difference-weights of the same element of X denoted by $\pm x''_{IJ}$ in the former and by $\pm x_{ij}$ in the present case, we have

$$k=k''+\xi_2, \qquad \text{and in particular} \quad \xi_0=\xi_0''+\xi_2.$$

Substituting for I, U, V, W, J, k'', $''$ the values given by these equations we see that if $\pm x_{ij}$ is an element of X lying in one of the ξ effective parametric diagonal lines of X_0, *i.e.* a non-zero element of X, the possible values of i and j, the possible values of the difference-weight $k=j-i$, and the possible values of i and j when k is given are those consistent with the respective sets of conditions

$$i\not<1, \ i\not>m-\xi_1, \ j\not<1+\xi_2, \ j\not>n, \ j-i\not<n-\rho\ ; \qquad \ldots \quad (6)$$

$$k\not<\xi_0, \ k\not>n-1\ ; \qquad \ldots \quad (6')$$

$$i\not<1, \ i\not<1+\xi_2-k, \ i\not>n-k, \ i\not>m-\xi_1\ ; \ j=i+k\ ; \qquad \ldots \quad (6'')$$

and we have further the following theorem :

Theorem III. *The non-zero elements of X (those forming the ξ effective parametric diagonal lines of X_0) are given by the general formula*

$$x_{i,j}=\Sigma a_{i,u}\, b_{u,v}\, c_{v,w}\, d_{w,j}, \qquad (j-i=k). \qquad \ldots \quad (e)$$

where the possible values of i, j and k are given by (6), (6'), (6'') and where for a given pair of values of i and j the summation extends over all distinct sets of values of u, v, w, which satisfy the conditions

$$u-u_2 \not< 1,\ u-u_2 \not> p_0\ ;\ v-P_2 \not< 1,\ v-P_2 \not> q_0\ ;\ w-Q_2 \not< 1,\ w-Q_2 \not> r_0,$$

or

$$u \not< 1+u_2,\ u \not> p-Q_1\ ;\ v \not< 1+P_2,\ v \not> q-R_1\ ;\ w \not< 1+Q_2,\ w \not> r-\delta_1,$$

and also make every term on the left positive in the equation

$$\{(u-i)-\alpha_0\}+\{(r-u)-\beta_0\}+\{(u-r)-\gamma_0\}+\{(j-w)-\delta_0\}$$
$$=k-(u-p).$$

Thus every element of the product matrix X is known.

8. Alternative method of evaluating the product matrix X of Art. 7 when it is not a zero matrix.

We will again suppose that A, B, C, D are all of the natural type $\{\pi, \pi\}$ and are expressed in their direct standard forms. The letters which have already been introduced will have the same meanings as before, and to show more clearly the general laws, we will introduce the new letters m_2, M_2, u_2, N_2, m_1, M_1, u_1, N_1 which are defined below.

If we pay regard to the rows which were cancelled in Art. 7, but do not perform any of the cancellations, we see from Lemmas A and B of Art. 6 that when the discriminators of the products AB, BC, CD, ABC, BCD, ABCD are all greater than 0, we can represent X as a product of hemipteric matrices of the first type in the form

$$X=ABCD=\begin{pmatrix} u_2, & p_0, & Q_1 \\ m_0, & p_1, & a_1 \end{pmatrix} \quad \begin{pmatrix} \beta_2, & p_2, & q_0, & R_1 \\ a_2, & p_0, & q_1, & \beta_1 \end{pmatrix}$$

$$\begin{pmatrix} \gamma_2, & q_2, & r_0, & \delta_1 \\ P_2, & q_0, & r_1, & \gamma_1 \end{pmatrix} \quad \begin{pmatrix} \delta_2, & r_2, & u_0 \\ Q_2, & r_0, & \delta_1 \end{pmatrix}, \qquad \ldots \quad (F)$$

where m_2, M_2, p_2, P_2, q_2, Q_2, r_2, R_2, u_2, N_2 are the positive integers defined by the statements that :

$m_2=0$;

$M_2=m_2+u_2=u_2$;

$P_2=p_2+\beta_2$ is the greater of M_2, $M_2+(q-\beta)$, *i.e.* the greater of

$$\beta_2,\ a_2+\beta_2-\beta_{20},$$

or

$$\rho_{\gamma\delta}-\{\rho_{\beta\gamma\delta}-\beta_{20}\},\ \rho_{\gamma\delta}-\{\rho_{\alpha\beta\gamma\delta}-a_{20}\};$$

$Q_2=q_2+\gamma_2$ is the greater of γ_2, $P_2+(r-\gamma)$, *i.e.* the greatest of

$$\gamma_2,\ \beta_2+\gamma_2-\gamma_{20},\ a_2+\beta_2+\gamma_2-\beta_{20}-\gamma_{20},$$

or

$$\delta-\{\rho_{\gamma\delta}-\gamma_{20}\},\quad \delta-\{\rho_{\beta\gamma\delta}-\beta_{20}\};$$

$R_2=r_2+\delta_2=\xi_2$ is the greater of δ_2, $Q_2+n-\delta$, *i.e.* the greatest of

$$\delta_2,\ \gamma_2+\delta_2-\delta_{20},\ \beta_2+\gamma_2+\delta_2-\gamma_{20}-\delta_{20},\ a_2+\beta_2+\gamma_2+\delta_2-\beta_{20}-\gamma_{20}-\delta_{20},$$

or

$$n-\{\delta-\delta_{20}\},\ n-\{\rho_{\gamma\delta}-\gamma_{20}\},\ n-\{\rho_{\beta\gamma\delta}-\beta_{20}\},\ n-\{\rho_{\alpha\beta\gamma\delta}-a_{20}\};$$

$N_2=n_2=R_2=\xi$;

where n_1, N_1, r_1, R_1, q_1, Q_1, p_1, P_1, m_1, M_1, are the positive integers defined by the statements that :

$n_1=0$;

$N_1=n_1+\delta_1=\delta_1$;

$R_1=r_1+\gamma_1$ is the greater of γ_1, $(q+\gamma)+N_1$, *i.e.* the greater of

$$\gamma_1,\quad \gamma_1+\delta_1-\gamma_{10},$$

or

$$\rho_{\alpha\beta}-\{\rho_{\alpha\beta\gamma}-\gamma_{10}\},\ \rho_{\alpha\beta}-\{\rho_{\alpha\beta\gamma\delta}-\delta_{10}\};$$

$Q_1=q_1+\beta_1$ is the greater of β_1, $(p-\beta)+R_1$, *i.e.* the greatest of

$$\beta_1,\ \beta_1+\gamma_1-\beta_{10},\ \beta_1+\gamma_1+\delta_1-\beta_{10}-\gamma_{10},$$

or

$$a-\{\rho_{\alpha\beta}-\beta_{10}\},\ a-\{\rho_{\alpha\beta\gamma}-\gamma_{10}\},\ a-\{\rho_{\alpha\beta\gamma\delta}-\delta_{10}\};$$

$P_1=p_1+a_1=\xi_1$ is the greater of a_1, $(m-a)+Q_1$, *i.e.* the greatest of

$$a_1,\ a_1+\beta_1-a_{10},\ a_1+\beta_1+\gamma_1-a_{10}-\beta_{10},\ a_1+\beta_1+\gamma_1+\delta_1-a_{10}-\beta_{10}-\gamma_{10},$$

or

$$m-\{a-a_{10}\},\ m-\{\rho_{\alpha\beta}-\beta_{10}\},\ m-\{\rho_{\alpha\beta\gamma}-\gamma_{10}\},\ m-\{\rho_{\alpha\beta\gamma\delta}-\delta_{10}\};$$

$M_1=m_1=P_1=\xi_1$;

and where $m_0=m-\xi_1$, p_0, q_0, r_0, $n=n-\xi_2$ have the values given in Art. 7.

From Lemma A of Art. 6 it will be seen that the constituents

$$\begin{pmatrix} p_2 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix} q_2 \\ \mathrm{P}_2 \end{pmatrix}, \quad \begin{pmatrix} r_2 \\ \mathrm{Q}_2 \end{pmatrix}$$

exist respectively when and only when $p_2 > 0$, $q_2 > 0$, $r_2 > 0$, and that they are then general non-zero simple post-major slopes with zero horizontal and vertical nullities having the non-zero parametric ranks a_2, P_2, Q_2. When $a_2 = 0$, then also $p_2 = 0$; when $\mathrm{P}_2 = 0$, then also $q_2 = 0$; when $\mathrm{Q}_2 = 0$, then also $r_2 = 0$. In fact

$$p_2, \quad q_2, \quad r_2$$

are respectively the greater of

$$0, a_2 - \beta_{2,0}; \quad 0, \mathrm{P}_2 - \gamma_{2,0}; \quad 0, \mathrm{Q}_2 - \delta_{2,0}.$$

From Lemma B of Art. 6 it will be seen that the constituents

$$\begin{pmatrix} \delta_1 \\ r_1 \end{pmatrix}, \quad \begin{pmatrix} \mathrm{R}_1 \\ q_1 \end{pmatrix}, \quad \begin{pmatrix} \mathrm{Q}_1 \\ p_1 \end{pmatrix}$$

exist respectively when and only when $r_1 > 0$, $q_1 > 0$, $p_1 > 0$, and that they are then general non-zero simple pre-major slopes with zero horizontal and vertical nullities having the non-zero parametric ranks δ_1, R_1, Q_1. When $\delta_1 = 0$, then also $r_1 = 0$; when $\mathrm{R}_1 = 0$, then also $q_1 = 0$; when $\mathrm{Q}_1 = 0$, then also $p_1 = 0$. In fact

$$r_1, \quad q_1, \quad p_1$$

are respectively the greater of

$$0, \delta_1 - \gamma_{1,0}; \quad 0, \mathrm{R}_1 - \beta_{1,0}; \quad 0, \mathrm{Q}_1 - a_{1,0}.$$

In the formula (E) we can strike out in succession

a_2 in A, a_2 in B : β_2 and p_2 in B, P_2 in C;

γ_2 and q_2 in C, Q_2 in D;

δ_1 in D, δ_1 in C; γ_1 and r_1 in C, R_1 in B;

β_1 and q_1 in B, Q_1 in A;

to represent in each case cancellations of corresponding passive rows. Then X is reduced to the form (D) of Art. 7, and the evaluation can be completed as before.

9. Generalisations and applications.

It will be immediately obvious what the general theorems for commutantal products of any number of factors are which correspond to the theorems given in Arts. 4, 5 and 7, and that those general

theorems can be either proved directly, or proved by induction when they have been proved for products of two factors. The results obtained are of very great generality, and are applicable, with the appropriate simplifications, to many special kinds of simple matrices, both non-ruled and ruled, the most obvious applications being to simple slopes.

The direct standard forms have been used throughout, but all results obtained for the commutantal product

$$[a]_m^p \; [b]_p^q \; [c]_q^r \; [d]_r^n = [x]_m^n \qquad \text{or} \qquad \text{ABCD}=\text{X} \quad \dots \quad (1)$$

remain valid for the commutantal product

$$\lceil d \rfloor_n^r \; \lceil c \rfloor_r^q \; \lceil b \rfloor_q^p \; \lceil a \rfloor_p^m = \lceil x \rfloor_n^m \qquad \text{or} \qquad \text{DCBA}=\text{X}, \quad \dots \quad (2)$$

with the proviso that $\alpha_2, \beta_2, \dots \xi_2$ are then the horizontal and $\alpha_1, \beta_1 \dots \xi_1$ the vertical nullities of A, B,...X in (2) ; for we can regard (2) as the conjugate of (1).

When the inverted standard forms are regarded as fundamental, all the results obtained for the product (1) can be converted into the corresponding results for the product

$$\lceil a \rfloor_m^p \; \lceil b \rfloor_p^q \; \lceil c \rfloor_q^r \; \lceil d \rfloor_r^n = \lceil x \rfloor_m^n \qquad \text{or} \qquad \text{ABCD}=\text{X} \quad \dots \quad (3)$$

by replacing

$$m, p, q, r, n \;:\; i, u, v, w, j \;;\; \alpha_1, \alpha_2 \;:\; \alpha_{10}, \alpha_{20} \;\; \dots$$

by

$$n, r, q, p, m \;:\; j, w, v, u, i \;:\; \alpha_2, \alpha_1 \;;\; \alpha_{20}, \alpha_{10} \;;\; \dots$$

Then every letter has the same meaning for the product (3) as it had for the product (1), $\alpha_1, \beta_1, \dots \xi_1$ being horizontal nullities, and α_1, $\beta_2 \dots \xi_2$ being vertical nullities. Thus the general formula for a parameter x_{ij} of X becomes

$$x_{ji} = \sum a_{ui} \, b_{vu} \, c_{wv} \, d_{jw} = \sum d_{jw} \, c_{wv} \, b_{vu} \, a_{ui} \; (i-j=k).$$

the possible values of i, j and k, and the range of the summation being as before. The only difference is that in passing from (1) to (3) the words 'initial' and 'final' must be interchanged in speaking of the horizontal and vertical parametric rows.

A note on the construction by points of a trinodal quartic.

By

Girindralal Mukherjee

1. In this note I propose to discuss a solution of the problem :—

To construct by *points* a trinodal quartic, of which the three nodes and five other points are given.

The solution is based upon two theorems which may be stated as follows :—

I. A, B, C are three given points and S is a given conic, Through one of these points, say A, two lines AP and AX are drawn meeting BC in P and X respectively and making with AB and AC a pencil of constant anharmonic ratio such that A (B, C, P, X)=k. If now from the foot of one of these lines (say X of the line AX) two tangents are drawn to the conic S, they will meet the other line AP in a pair of points P', P'' which will trace out a trinodal quartic, having A, B, P for nodes and touching S at four points.

II. A, B, C are the three nodes of a trinodal quartic U. Through one of the nodes, say A, two lines AP and AX are drawn, meeting BC in P and X respectively and making with AB, AC a pencil of constant anharmonic ratio such that A (B, C, P, X)=k. If now one of these lines, say AP, meet the curve U in the two points P', P'' ; then the lines joining X (the foot of the other line AX) to these points will envelop a conic S, having quartic contact with U.

The following few points may be noted in connection with these theorems :—

(*a*) The lines AB and AX trace out two pencils which are concentric and homographic, having AB, AC, for their common self-corresponding elements.

The pencil traced out by AP will in future be referred to as the *first* pencil ; and that traced out by AX as the *second* pencil.

(*b*) Let U be the quartic traced out by the points P', P'' in theorem I. It can be easily seen that if BQ and BY be drawn through B

meeting AC in Q and Y respectively and making with BC and BA a pencil of which the anharmonic ratio is $1-\frac{1}{k}$; that is to say $B(C, A, Q, Y)=1-\frac{1}{k}$, then BQ will meet the tangents drawn to S from Y in a pair of points which will lie on U. Similarly if through C, CR and CZ be drawn meeting AB in R and Z respectively such that $C\ (A, B, R, Z)=\frac{1}{1-k}$; then CR will meet the tangents to S from Z in a pair of points lying on U.

2. From what has been said above it is clear that the first step in the solution of the problem under consideration will be to determine the conic S. Now the conic S can be determined in the following manner :—

Let A, B, C be the three nodes and P_1, P_2, P_3, P_4, P_5 be the five other given points. Join one of the points A, B, C (say A) to each of the five given points P. Drawn AX_1, AX_2, AX_3, AX_4, AX_5 meeting BC in X_1, X_2, X_3, X_4, X_5 respectively such that

$$A\ (B, C, P_1, X_1)=A\ (B, C, P_2, X_2)=\ldots=k.$$

Drawn a conic touching the five lines P_1X_1, $P\ X_2$, $P\ X_3$, P_4X_4, P_5X_5 ; this conic will be the required conic S. Having determined S, the next step will be to determine the points where lines drawn through either A, B or C meet the tangents drawn to this conic S from the feet of the corresponding lines as explained above. These points will be points on the quartic U to be drawn.

It is easily seen that in solving the problem we require the help of the ruler and compasses only. Hence the problem of constructing a trinodal quartic of which the three nodes and five other points are given may be regarded as a problem of the second degree.

3. I shall show that the curve U generated above is a trinodal quartic curve, having A, B, C for nodes ; or in other words that it possesses all the following properties of a trinodal quartic, *viz.*—

(1) that the order of the curve is four ;

(2) that it has got three nodes and A, B, C are these nodes ;

(3) that it has *four* double tangents ;

(4) that it has *six* inflexional points ;

(5) that the class of the curve is six.

(1) The degree of a curve can be ascertained by determining the number of its intersections with a right line. Let L be any right line and it is required to determine the number of its intersections with U. Let L meet BC in O and AB, AC, AP_1, AP_2, AP_3, AP_4, AP_5 in B', C', P'_1, P'_2, P'_3, P'_4, P'_5 respectively ; then it is easily seen that the two ranges of points O (B, C, X_1, X_2, X_3, X_4, X_5) and O (B', C', P'_1, P'_2, P'_3, P'_4, P'_5) are homographic but not in perspective. Hence the lines BB', CC', P'_1X_1, P'_2X_2, etc. joining corresponding points of the two hamographic ranges will envelop a conic. Let C be this conic ; it will evidently also touch the bases L and BC. Now, if L_1 be a point common to L and U and M_1 be the point on BC such that A (B, C, L_1, M_1)=A (B, C, P'_1, X_1) $=k$; then L_1M_1 will touch both C and S. Hence L_1 to be a common points of intersection of L and U must also lie on a common tangent of C and S. As there are four common tangents to C and S ; the number of points common to L and U must also be *four* and these points are the four points of intersection of L with the four common tangents of C and S. Thus it is proved that the degree of the curve is four.

(2) That A, B, C are double points on U can be easily seen by considering the intersections with U of the lines AB, AC, BC.

Again let any line AP through A meet U in the points P', P'' and let X be a point on BC, such that

$$A\ (B, C, P', X)=A\ (B, C, P_1, X_1)=k$$

then XP', XP'' will be tangents to S. If now AP'P'' be made to rotate round A until P' moves upto and ultimately coincides with A ; then AP'' will be a tangent to the curve at A. But then XP' becomes coincident with AX, one of the tangents from A to S. Thus a tangent to the curve U at A is a ray of the first pencil corresponding to a tangent to S from A, regarded as a ray of the second pencil. As there can be drawn two tangents to S from A, it at once follows that there can be drawn two tangents to the curve U at A, and they are the rays of the first pencil corresponding to the tangents to S from A, considered as rays of the second pencil.

Similarly also it can be shown that two tangents can be drawn to U at each of the points B and C and they are lines corresponding to the tangents drawn from them respectively to S.

In this way also it is seen that A, B, C are double points on the curve U.

The cases when these tangents will be real, imaginary or coincident can be easily distinguished.

(3) If in (1) L be a double tangent to the curve, C will have double contact with S. Hence the number of double tangents to the curve is evidently equivalent to the number of such conics C that can be drawn so as to touch the three given lines BC, CA, AB and have double contact with the given conic S. The number of such conics, as we know, is four and consequantly the number of bitangents to the curve is *four*.

Let S intersect the sides BC, CA, AB of the triangle ABC in the three pairs of points A_1, A_2, B_1, B_2 ; C_1, C_2 respectively in the order in which they are written ; then it can be shown that the four Pascal lines

$$\begin{array}{lll}
C_2A_2B_2C_1A_1B_1 & C_2A_1B_2C_1A_2B_1 & C_2A_1B_1C_1A_2B_2 \\
\overbrace{\quad\quad} & \overbrace{\quad\quad} & \overbrace{\quad\quad} \\
C_2A_2\cdot C_1A_1, & C_2A_1\cdot C_1A_2, & C_2A_1\cdot C_1A_2, \\
A_2B_2\cdot A_1B_1, & A_1B_2\cdot A_2B_1, & A_1B_1\cdot A_2B_2, \\
B_2C_1\cdot B_1C_2. & B_2C_1\cdot B_1C_2, & B_1C_1\cdot B_2C_2,
\end{array}$$

$$\begin{array}{l}
C_1A_1B_2C_2A_2B_1 \\
\overbrace{\quad\quad} \\
C_1A_1\cdot C_2A_2, \\
A_1B_2\cdot A_2B_1, \\
B_2C_2\cdot B_1C_1.
\end{array}$$

(where the Pascal line $C_2A_2B_2C_1A_1B_1$ is obtained by joining the three points of intersection of the three pairs of lines C_2A_2, C_1A_1 ; A_2B_2, A_1B_1 ; B_2C_1, B_1C_2 respectively) are the four common chords of contact of the conics C and S.

(4) If L be an inflexional tangent to the curve U at any point L_1 ; then C will have contact of the second order with S. Hence the number of inflexional tangents, and consequently of inflexional points on the curve, will be the same as the number of conics C that

can be drawn touching three given lines and osculating a given conic. As the latter number is six, it follows that the number of *points of inflexion* on the curve is at most *six*.

(5) If L touches the curve U at L_1 ; then C and S will touch one another ; and if M_1 be the point on BC such that

$$A\ (B, C, L_1, M_1) = A\ (B, C, P_1, X_1) = k$$

then $L_1\ M_1$ will touch both C and S at their common points of contact. Therefore L will be the second tangent drawn from L_1 to C.

If L passes through a point P which is not on the curve U and M be a point on BC such that

$$A\ (B, C, P, M) = A\ (B, C, P_1, X_1) = k$$

then PM will touch C. Hence when P is given, to determine PL_1 we have to draw the conic C of which it is a conic touching four given lines (BC, CA, AB, PM) and a given conic S. As the problem of determining C admits of six different solutions, it follows that *six* tangents can be drawn to the curve U from P. In other words the *class* of the curve is *six*.

When P is on the curve, PM touches S also ; and in this case two of the six solutions become identical and this double solution will then correspond to the tangent to the curve at P. Thus when P is on the curve, besides the tangent at P, there can be drawn to the curve four other tangents from it.

If P coincides with any one of the points A, B or C, this construction will fail ; and in that case it can be shown that only two tangents can be drawn to the curve from each of these points. By making AP in sub-sec. (2) rotate round A until P″ moves up to and coincides with P′, it can be shown that the tangents from A to the curve will be rays of the *first* pencil corresponding to the lines AA_1, AA_2 considered as rays of the *second* pencil (A_1, A_2 being the points of intersection of BC with S).

Similarly the tangents from B and C can be determined.

4. I have shown that the curve U possesses all the Pluckerian characteristics of a trinodal quartic. It now remains to show that the curve thus generated is a *unique* one ; for it may be argued that by varying k, different curves satisfying the given conditions can be drawn. But upon examination this will be found not to be the case ;

for if it were possible to draw two different curves by giving two different values to k, then we would have the absurdity of two quartic curves intersecting in $(3\times4+5)=$ seventeen or more than sixteen, points.

5. I conclude this note by giving a few additional problems on the trinodal quartic, the solutions of which can be made to depend upon the two theorems stated in Art. 1. The problems with their solutions are given below :—

I. Given the three nodes A, B, C ; one other point P and the conic S ; construct the quartic.

Here the first thing for determination is the constant of anharmonic ratio (k). If the two tangents from P to S meet BC in X_1 and X_2 ; then k will be determined either by A (B, C, P, X_1) or A (B, C, P, X_2). Hence there will be two solutions.

II. Given the three other poinis P_1, P_2, P_3 and a bitangent L : to construct the quartic.

Here the conic S is to be first determined. This can be done in the following manner :—Take the points X_1, X_2, X_3 on BC such that A (B, C, P_1, X_1)=A (B, C, P_2, X_2)=A (B, C, P_3, X_3). Let AP_1, AP_2, AP_3, meet L in P'_1, P'_2, P'_3 respectively. Draw a conic C touching BC, CA, AB, L and $P'_1X'_1$, which will also touch P'_2X_2, P'_3X_3. Draw another conic (S) touching the lines P_1X_1, P_2X_2, P_3X_3 and having double contact with C. This conic will be the required conic S.

This problem admits of four different solutions.

III. Given the three double points A, B, C ; one other point P and two bitangents L_1 and L_2 : to construct the quartic.

Here the conic S can be determined in the following manner :—

Take any point X on BC ; and let P_1, P_2 be the points where AP meet L_1, L_2 respectively. Draw a conic C_1 touching BC, CA, AB, L_1, P_1X and another conic C_2 touching BC, CA, AB, L_2 and P_2X. Draw a conic (S) touching PX and having double contant with both C_1 and C_2. This last-drawn conic will be the required conic S.

This problem also, like the preceding two admits of more than one solution.

On Liquid Motion inside Rotating Arcs of Three and Four Confocal Parabolas

By

N. M. Basu

The problem of determining the two-dimensional motion of a homogeneous incompressible liquid inside rotating cylinders has engaged the attention of many eminent mathematicians, but up to now the problem has been solved for only a small number of cases. The present state of analysis does not enable us to attack with success the general problem for any arbitrary boundary. A solution for the case in which the cross-section of the rotating cylinder consists of arcs of two confocal parabolas has been given by Ferrers.[1] The object of the present paper is to present the solution for two new cases, *viz.*, (1) when the cross-section consists of *three confocal parabolas* and (2) when the cross-section consists of *four confocal parabolas*, all belonging to the same orthogonal system.

It may be mentioned that the hydrodynamical solutions given in this paper also furnish the solutions of the torsion problem in the Theory of Elasticity for the same boundaries.

§ 1. *Preliminary Observations.*

Take the common focus of the parabolic arcs of a cross-section as the origin and the common axis as the axis of x. If λ and μ be the parameters of the two parabolas which pass through the point (x, y), we have

$$x=\lambda-\mu,$$

$$y^2=4\lambda\mu,$$

λ being the parameter of the parabola whose axis is the negative x-axis.

Let $\lambda=\xi^2$ and $\mu=\eta^2$,

so that we have

$$\left.\begin{aligned} x&=\xi^2-\eta^2 \\ y&=\pm 2\xi\eta \end{aligned}\right\} \quad \ldots \quad (1)$$

[1] Ferrers: "On the motion of water contained in certain cylindrical vessels, &c.," *Quarterly Journal*, Vol. 17.

To avoid ambiguity we shall suppose that η is *always positive* while ξ may have *either sign*, and instead of (1) we shall write

$$\left.\begin{aligned} x&=\xi^2-\eta^2,\\ y&=2\xi\eta. \end{aligned}\right\} \quad \ldots \quad (2)$$

Thus η=constant are complete parabolas and ξ=constant are half-parabolas, the upper halves corresponding to positive values and the lower halves to negative ones.

It is easily seen from (2) that ξ and η are conjugate functions given by the relation

$$x+iy=(\xi+i\eta)^2.$$

Hence the stream function ψ must satisfy the condition

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}=0 \quad \ldots \quad (3)$$

at all points occupied by the liquid.

Also, if the cylinder rotates with angular velocity ω about an axis through the focus, we must have

$$\psi=\tfrac{1}{2}\omega(x^2+y^2)+\text{constant} \quad \ldots \quad (4)$$

at all points on the boundary.

Now from (2) we have

$$x^2+y^2=(\xi^2+\eta^2)^2=(\xi^4-6\xi^2\eta^2+\eta^4)+8\xi^2\eta^2.$$

Since $\xi^4-6\xi^2\eta^2+\eta^4$ satisfies the equation (3), we shall assume

$$\psi=\tfrac{1}{2}\omega(\xi^4-6\xi^2\eta^2+\eta^4)+\psi_1.$$

Then ψ_1 will have to satisfy the conditions

$$\frac{\partial^2\psi_1}{\partial\xi^2}+\frac{\partial^2\psi_1}{\partial\eta^2}=0 \quad \ldots \quad (5)$$

at all points occupied by the liquid, and

$$\psi_1=4\omega\xi^2\eta^2+\text{constant} \quad \ldots \quad (6)$$

at all points of the boundary.

§ 2. *Three Confocal Parabolas.*

Let the cylinder be bounded by the three confocal parabolic cylinders $\eta=\eta_1$, $\eta=\eta_2$ and $\xi=\pm\xi_1$.

Let $\eta_2>\eta_1$, $\eta_2-\eta_1=2\beta$ and $\eta_2+\eta_1=2\delta$, so that $\eta_2=\delta+\beta$, and $\eta_1=\delta-\beta$.

Assume

$$\psi_1=\sum_0^\infty \cos(2n+1)\frac{\pi\xi}{2\xi_1}\left[A_n\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)\right.$$

$$\left.+B_n\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)\right]$$

$$+\sum_0^\infty C_n\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\cosh(2n+1)\frac{\pi\xi}{2\beta}.$$

The equation (5) is obviously satisfied. The equation (6) will be satisfied if

$$\psi_1=4\omega\xi^2\eta_1^2 \text{ when } \eta=\eta_1 \quad \text{and } -\xi_1<\xi<\xi_1$$

$$\psi_1=4\omega\xi^2\eta_2^2 \text{ when } \eta=\eta_2 \quad \text{and } -\xi_1<\xi<\xi_1,$$

and $$\psi_1=4\omega\xi_1^2\eta^2 \text{ when } \xi=\pm\xi_1 \text{ and } \eta_1<\eta<\eta_2.$$

Hence, putting $\eta=\eta_2$, we must have, for all values of ξ between $-\xi_1$ and ξ_1,

$$4\omega\xi^2\eta_2^2=\sum_0^\infty A_n\cos(2n+1)\frac{\pi\xi}{2\xi_1}\sinh\frac{2n+1)\pi\beta}{\xi_1}.$$

Multiplying both sides by $\cos 2n+1)\frac{\pi\xi}{2\xi_1}\,d\xi$ and integrating between the limits $-\xi_1$ and ξ_1, we get

$$\xi_1 A_n\sinh\frac{(2n+1)\pi\beta}{\xi_1}=4\omega\eta_2^2\int_{-\xi_1}^{\xi_1}\xi^2\cos(2n+1)\frac{\pi\xi}{2\xi_1}\,d\xi$$

$$=\frac{64\omega\xi_1^3\eta_2^2}{\pi^3}\int_0^{\frac{\pi}{2}} z^2\cos(2n+1)z\,dz$$

$$=\frac{(-1)^n 64\omega\xi_1^3\eta_2^2}{(2n+1)\pi}\left[\frac{1}{4}-\frac{2}{(2n+1)^2\pi^2}\right]$$

$$\therefore\ A_n=\frac{(-1)^n}{(2n+1)\pi}\cdot\frac{64\omega\xi_1^2\eta_2^2\left[\frac{1}{4}-\frac{2}{(2n+1)^2\pi^2}\right]}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}.$$

Similarly

$$B_n = \frac{(-1)^n}{(2n+1)\pi} \cdot \frac{64\omega\xi_1^2\eta_1^2\left[\frac{1}{4} - \frac{2}{(2n+1)^2\pi^2}\right]}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}.$$

To find C_n put $\xi = \pm\xi_1$. Then we must have, for all values of η between η_1 and η_2,

$$4\omega\xi_1^2\eta^2 = \sum_0^\infty C_n \cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\cosh\frac{(2n+1)\pi\xi_1}{2\beta}.$$

Multiplying both sides by $\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)d\eta$ and integrating between the limits η_1 and η_2, we obtain

$$\beta C_n \cosh\frac{(2n+1)\pi\xi_1}{2\beta} = 4\omega\xi_1^2\int_{\eta_1}^{\eta_2}\eta^2\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)d\eta$$

$$= 4\omega\xi_1^2\int_{-\beta}^{\beta}(z^2+2z\delta+\delta^2)\cos\frac{(2n+1)\pi z}{2\beta}dz.$$

Now $$\int_{-\beta}^{\beta} z^2\cos\frac{(2n+1)\pi z}{2\beta}dz = \frac{(-1)^n 4\beta^3}{(2n+1)\pi}\left[1-\frac{8}{(2n+1)^2\pi^2}\right]$$

$$\int_{-\beta}^{\beta} z\cos\frac{(2n+1)\pi z}{2\beta}dz = 0,$$

$$\int_{-\beta}^{\beta}\cos\frac{(2n+1)\pi z}{2\beta}dz = \frac{(-1)^n\cdot 4\beta}{(2n+1)\pi}.$$

Thus $$C_n = \frac{(-1)^n 16\omega\xi_1^2}{(2n+1)\pi}\cdot\frac{\left[(\beta^2+\delta^2)-\frac{8\beta^2}{(2n+1)^2\pi^2}\right]}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}.$$

We thus have

$$\psi=\tfrac{1}{2}\omega(\xi^4-6\xi^2\eta^2+\eta^4)+\psi_1,$$

where

$$\psi_1=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}\cos\frac{(2n+1)\pi\xi}{2\xi_1}$$

$$\times\left[\frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right]$$

$$+16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left\{(\beta^2+\delta^2)-\frac{8\beta^2}{(2n+1)^2\pi^2}\right\}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)$$

$$\times\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}$$

$$=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left[\frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right.$$

$$\left.\times\cos\frac{(2n+1)\pi\xi}{2\xi_1}+(\beta^2+\delta^2)\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right]$$

$$-128\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\left[\frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}\right.$$

$$\left.\times\cos\frac{(2n+1)\pi\xi}{2\xi_1}+\beta^2\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right].$$

Remembering that $\phi+i\psi$ is a function of $\xi+i\eta$, the expression for the velocity potential ϕ can be readily worked out.

We thus obtain

$$\phi=-2\omega\xi\eta(\xi^2-\eta^2)+\phi_1,$$

where

$$\phi_1=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\Bigg[\frac{\eta_2^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}$$

$$\times\sin\frac{(2n+1)\pi\xi}{2\xi_1}-(\beta^2+\delta^2)\frac{\sinh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\Bigg]$$

$$-128\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\Bigg[\frac{\eta_2^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)+\eta_1^2\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{\xi_1}}$$

$$\times\sin\frac{(2n+1)\pi\xi}{2\xi_1}-\beta^2\frac{\sinh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\Bigg].$$

§ 3. *Simplified Results.*

The results obtained in the preceding article can be simplified to a certain extent in the following manner.

Substituting the values of η_1 and η_2 in terms of β and δ and observing that

$$\sinh\frac{(2n+)\pi}{2\xi_1}(\eta-\eta_1)+\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)$$

$$=2\sinh\frac{(2n+1)\pi\beta}{2\xi_1}\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta),$$

$$\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\eta_1)-\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta_2-\eta)$$

$$=2\cosh\frac{(2n+1)\pi\beta}{2\xi_1}\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta),$$

and $$\sinh\frac{(2n+1)\pi\beta}{\xi_1}=2\sinh\frac{(2n+1)\pi\beta}{2\xi_1}\cosh\frac{(2n+1)\pi\beta}{2\xi_1},$$

we find that ψ_1 can be written as

$$\psi_1=16\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\{(\beta^2+\delta^2)(P+Q)+2\beta\delta R\}$$

$$-128\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\{(\beta^2+\delta^2)P+\beta^2Q+2\beta\delta R\}, \quad \ldots \quad (7)$$

where $$P=\frac{\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\xi_1}}\cos\frac{(2n+1)\pi\xi}{2\xi_1}, \quad \ldots \quad (8)$$

$$Q=\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta), \quad \ldots \quad (9)$$

and $$R=\frac{\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\xi_1}}\cos\frac{(2n+1)\pi\xi}{2\xi_1}. \quad \ldots \quad (10)$$

Now it is obvious that, when $\eta=\eta_1$ and $-\xi_1<\xi<\xi_1$,

$$\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}(P+Q)=\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\cos\frac{(2n+1)\pi\xi}{2\xi_1}=\tfrac{1}{4}.$$

In a similar manner it will be found that, when $\eta=\eta_2$ and $-\xi_1<\xi<\xi_1$ or when $\xi=\pm\xi_1$ and $\eta_1<\eta<\eta_2$,

$$\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}(P+Q)=\tfrac{1}{4}.$$

Thus the above equation holds for all points on the boundary.

Also $$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\left\{\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}(P+Q)\right\}=0.$$

Hence, everywhere within the cylinder we must have

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)\pi}(P+Q)=\tfrac{1}{4}. \qquad \ldots \quad (11)$$

Integrating the equation (11) with respect to η, we have

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}\left\{2\xi_1\frac{\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\xi_1}}\cos\frac{(2n+1)\pi\xi}{2\xi_1}\right.$$

$$\left.+2\beta\frac{\cosh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\right\}=\frac{\eta}{4}+f(\xi).$$

To determine $f(\xi)$ put $\eta=\delta$ in both sides. Then the lefthand side vanishes.

Thus $$f(\xi)=-\frac{\delta}{4}.$$

Substituting this value and integrating again with respect to η, we obtain

$$\sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}\left\{4\xi_1^2P-4\beta^2Q\right\}=\frac{(\eta-\delta)^2}{8}+F(\xi). \qquad \ldots \quad (12)$$

Since the lefthand side remains unchanged when $-\xi$ is written for ξ. $F(\xi)$ must be an even function of ξ.

To determine this function operate on both sides by $\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}$. Then the lefthand side vanishes and we have

$$F''(\xi)=-\tfrac{1}{2}.$$

$$\therefore \quad F(\xi)=-\tfrac{1}{8}\xi^2+k,$$

where k is a constant of integration.

To find the value of k substitute this value of $F(\xi)$ in (12) and put $\xi=\pm\xi_1$ and $\eta=\eta_1$ or η_2. Then the lefthand side disappears and we have

$$0=\tfrac{1}{8}(\beta^2-\xi_1^2)+k.$$

Thus $$k=\tfrac{1}{2}(\xi_1^2-\beta^2)$$

and $$\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}(\xi_1^2\mathrm{P}-\beta^2\mathrm{Q})=\tfrac{1}{32}\{(\eta-\delta)^2-\xi^2+\xi_1^2-\beta^2\}^2 \quad \ldots \quad (13)$$

Let $$\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\{(\beta^2+\delta^2)\mathrm{P}+\beta^2\mathrm{Q}\}=\mathrm{U}. \qquad (14)$$

From (13) and (14) we easily obtain

$$\mathrm{U}=-\tfrac{1}{32}\{(\eta-\delta)^2-\xi^2+\xi_1^2-\beta^2\}^2+(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\mathrm{P}; \qquad (15)$$

$$\mathrm{U}=\frac{(\beta^2+\delta^2)}{32\xi_1^2}\{(\eta-\delta)^2-\xi^2+\xi_1^2-\beta^2\}^2$$
$$+\frac{\beta^2}{\xi_1^2}(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\mathrm{Q}. \qquad (16)$$

From (7), (11) and (14) we have

$$\psi_1=4\omega\xi_1^2(\beta^2+\delta^2)-128\omega\xi_1^2\mathrm{U}+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}\mathrm{R}.$$

Using (15) or (16) we finally have

$$\psi_1=4\omega\xi_1^2\{(\eta-\delta)^2-\xi^2+\xi_1^2+\delta^2\}^2-128\omega\xi_1^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\mathrm{P}$$
$$+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}\mathrm{R},$$

$$\psi_1=4\omega(\beta^2+\delta^2)\{\xi^2-(\eta-\delta)^2+\beta^2\}^2-128\omega\beta^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\mathrm{Q}$$
$$+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}\mathrm{R},$$

where P, Q and R are given by the relations (8), (9) and (10).

The corresponding expressions for ϕ_1 may be easily deduced.

Thus

$$\phi_1=8\omega\xi_1^2\xi(\eta-\delta)-128\omega\xi_1^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\mathrm{P}'$$
$$+32\beta\delta\omega\xi_1^2\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}\mathrm{R}',$$

$$\phi_1=-8\omega'(\beta^2+\delta^2)\xi(\eta-\delta)+128\omega\beta^2(\beta^2+\delta^2+\xi_1^2)\sum_0^\infty \frac{(-1)^n}{(2n+1)^5\pi^5}Q'$$

$$+32\beta\delta\omega\xi_1^2\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\left\{1-\frac{8}{(2n+1)^2\pi^2}\right\}R',$$

where
$$P'=-\frac{\sinh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\xi_1}}\sin\frac{(2n+1)\pi\xi}{2\xi_1},$$

$$Q'=\frac{\sinh\frac{(2n+1)\pi\xi}{2\beta}}{\cosh\frac{(2n+1)\pi\xi_1}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta),$$

and
$$R'=\frac{\cosh\frac{(2n+1)\pi}{2\xi_1}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\xi_1}}\sin\frac{(2n+1)\pi\xi}{2\xi_1}.$$

§ 4. *Four Confocal Parabolas.*

Suppose the cylinder is bounded by the four confocal parabolic cylinders $\xi=\xi_1$, $\xi=\xi_2$, $\eta=\eta_1$ and $\eta=\eta_2$.

Let
$$\xi_2>\xi_1,\ \eta_2>\eta_1.$$

$$\xi_2-\xi_1=2\alpha,\ \eta_2-\eta_1=2\beta$$

and
$$\xi_2+\xi_1=2\gamma,\ \eta_2+\eta_1=2\delta,$$

so that
$$\xi_2=\gamma+\alpha,\ \xi_1=\gamma-\alpha$$

and
$$\eta_2=\delta+\beta,\ \eta_1=\delta-\beta.$$

Assume

$$\psi_1=\sum_0^\infty\cos\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma)\left[A_n\sinh\frac{(2n+1)\pi}{2\alpha}(\eta-\eta_1)\right.$$

$$\left.+B_n\sinh\frac{(2n+1)\pi}{2\alpha}(\eta_2-\eta)\right]$$

$$+\sum_0^\infty\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\left[C_n\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1)\right.$$

$$\left.+D_n\sinh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi)\right].$$

Then proceeding exactly as in the preceding articles we obtain the following results.

$$A_n = 16\omega\eta_2^2 \cdot \frac{(-1)^n}{(2n+1)\pi}\left[(a^2+\gamma^2) - \frac{8a^2}{(2n+1)^2\pi^2}\right], \Big/ \sinh\frac{(2n+1)\pi\beta}{2a},$$

$$C_n = 16\omega\xi_2^2 \cdot \frac{(-1)^n}{(2n+1)\pi}\left[(\beta^2+\delta^2) - \frac{8\beta^2}{(2n+1)^2\pi^2}\right], \Big/ \sinh\frac{(2n+1)\pi a}{\beta},$$

B_n and D_n are what A_n and C_n become when η_1^2 and ξ_1^2 are written for η_2^2 and ξ_2^2 respectively:

whence

$$\psi_1 = 16\omega\sum_0^\infty \frac{(-1)^n}{(2n+1)\pi}\Big[(a^2+\gamma^2)\cos\frac{(2n+1)\pi}{2a}(\xi-\gamma) \times \frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2a}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2a}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{2a}}$$

$$+(\beta^2+\delta^2)\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta) \times \frac{\xi_2^2\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1)+\xi_1^2\sinh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi)}{\sinh\frac{(2n+1)\pi a}{\beta}}\Big]$$

$$-128\omega\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}\Big[a^2\cos\frac{(2n+1)\pi}{2a}(\xi-\gamma) \times \frac{\eta_2^2\sinh\frac{(2n+1)\pi}{2a}(\eta-\eta_1)+\eta_1^2\sinh\frac{(2n+1)\pi}{2a}(\eta_2-\eta)}{\sinh\frac{(2n+1)\pi\beta}{2a}}$$

$$+\beta^2\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta)$$

$$\times\frac{\xi_2^2\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1)+\xi_1^2\sinh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi)}{\sinh\frac{(2n+1)\pi a}{\beta}}\Big]$$

and

$$\phi_1=16\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\Big[(\alpha^2+\gamma^2)\sin\frac{(2n+1)\pi}{2a}(\xi-\gamma)\ \frac{\eta_2^2Y_1+\eta_1^2Y_2}{\sinh\frac{(2n+1)\pi\beta}{a}}$$

$$-(\beta^2+\delta^2)\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\ \frac{\xi_2^2X_1+\xi_1^2X_2}{\sinh\frac{(2n+1)\pi a}{\beta}}\Big]$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\Big[\alpha^2\sin\frac{(2n+1)\pi}{2a}(\xi-\gamma)\ \frac{\eta_2^2Y_1+\eta_1^2Y_2}{\sinh\frac{(2n+1)\pi\beta}{a}}$$

$$-\beta^2\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta)\ \frac{\xi_2^2X_1+\xi_1^2X_2}{\sinh\frac{(2n+1)\pi a}{\beta}}\Big].$$

where $X_1=\cosh\frac{(2n+1)\pi}{2\beta}(\xi-\xi_1)$,

$X_2=\cosh\frac{(2n+1)\pi}{2\beta}(\xi_2-\xi)$.

$Y_1=\cosh\frac{(2n+1)\pi}{2a}(\eta-\eta_1)$,

and $Y_2=\cosh\frac{(2n+1)\pi}{2a}(\eta_2-\eta)$

Substituting the values of ξ_1, ξ_2, η_1 and η_2 in terms of a, β, γ and δ and adopting the method of the preceding articles, we get the following additional results :

$$\psi_1=16\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}\ [(\alpha^2+\gamma^2)(\beta^2+\delta^2)(P+Q)+2\beta\delta(\alpha^2+\gamma^2)R$$

$$+2\alpha\gamma(\beta^2+\delta^2)S]$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\alpha^2(\beta^2+\delta^2)P+\beta^2(\alpha^2+\gamma^2)Q+2\alpha^2\beta\delta R+2\beta^2\alpha\gamma S],$$

where

$$P=\frac{\cosh\frac{(2n+1)\pi}{2a}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2a}}\cos\frac{(2n+1)\pi}{2a}(\xi-\gamma), \quad \ldots \quad (17)$$

$$Q=\frac{\cosh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\cosh\frac{(2n+1)\pi a}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta), \quad \ldots \quad (18)$$

$$R=\frac{\sinh\frac{(2n+1)\pi}{2a}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2a}}\cos\frac{(2n+1)\pi}{2a}(\xi-\gamma), \quad \ldots \quad (19)$$

and

$$S=\frac{\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\sinh\frac{(2n+1)\pi a}{2\beta}}\cos\frac{(2n+1)\pi}{2\beta}(\eta-\delta), \quad \ldots \quad (20)$$

We shall also find that, at all points on and within the boundary,

$$\sum_0^\infty \frac{(-1)^n}{(2n+1)^3\pi^3}(P+Q)=\tfrac{1}{4}.$$

$$\sum_0^\infty \frac{(-1)^n}{(2n+1)^5\pi^5}[a^2P-\beta^2Q]=\tfrac{1}{32}[(\eta-\delta)^2-(\xi-\gamma)^2+a^2-\beta^2].$$

and if

$$\sum_0^\infty \frac{(-1)^n}{(2n+1)^5\pi^5}[a^2(\beta^2+\delta^2)P+\beta^2(a^2+\gamma^2)Q]=U.$$

then

$$U=-\frac{a^2+\gamma^2}{32}[(\eta-\delta)^2-(\xi-\gamma)^2+a^2-\beta^2]$$

$$+a^2(a^2+\beta^2+\gamma^2+\delta^2)\sum_0^\infty\frac{(-1)^n}{(2n+1)^5\pi^5}P,$$

or

$$U=\frac{\beta^2+\delta^2}{32}[(\eta-\delta)^2-(\xi-\gamma)^2+\alpha^2-\beta^2]$$

$$+\beta^2(\alpha^2+\beta^2+\gamma^2+\delta^2)\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\,Q.$$

Finally

$$\psi_1=4\omega(\alpha^2+\gamma^2)[(\eta-\delta)^2-(\xi-\gamma)^2+\alpha^2+\delta^2]$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R+\alpha\gamma(\beta^2+\delta^2)S]$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\alpha^2(\alpha^2+\beta^2+\gamma^2+\delta^2)P+2\alpha^2\beta\delta R+2\beta^2\alpha\gamma S],$$

or $$\psi_1=-4\omega(\beta^2+\delta^2)[(\eta-\delta)^2-(\xi-\gamma)^2-\beta^2-\gamma^2]$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R+\alpha\gamma(\beta^2+\delta^2)S]$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}[\beta^2(\alpha^2+\beta^2+\gamma^2+\delta^2)Q+2\alpha^2\beta\delta R+2\beta^2\alpha\gamma S],$$

where P, Q, R and S are given by (17), (18), (19) and (20),

and $$\phi_1=8\omega(\alpha^2+\gamma^2)(\xi\eta-\delta\xi-\gamma\eta)$$

$$+32\omega\sum_0^\infty\frac{(-1)}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R'-\alpha\gamma(\beta^2+\delta^2)S']$$

$$-128\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)^3\pi^3}\,\alpha^2(\alpha^2+\beta^2+\gamma^2+\delta^2)P'+2\alpha^2\beta\delta R'-2\beta^2\alpha\gamma S'],$$

or $$\phi_1=-8\omega(\beta^2+\delta^2)(\xi\eta-\delta\xi-\gamma\eta)$$

$$+32\omega\sum_0^\infty\frac{(-1)^n}{(2n+1)\pi}[\beta\delta(\alpha^2+\gamma^2)R'-\alpha\gamma(\beta^2+\delta^2)S']$$

$$+128\omega\sum_0^\infty\frac{(-1)}{(2n+1)^3\pi^3}[\beta^2(\alpha^2+\beta^2+\gamma^2+\delta^2)Q'-2\alpha^2\beta\delta R'+2\beta^2\alpha\gamma S']$$

where

$$P'=\frac{\sinh\frac{(2n+1)\pi}{2\alpha}(\eta-\delta)}{\cosh\frac{(2n+1)\pi\beta}{2\alpha}}\sin\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma),$$

$$Q'=\frac{\sinh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\cosh\frac{(2n+1)\pi\alpha}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta),$$

$$R'=\frac{\cosh\frac{(2n+1)\pi}{2\alpha}(\eta-\delta)}{\sinh\frac{(2n+1)\pi\beta}{2\alpha}}\sin\frac{(2n+1)\pi}{2\alpha}(\xi-\gamma),$$

and

$$S'=\frac{\cosh\frac{(2n+1)\pi}{2\beta}(\xi-\gamma)}{\sinh\frac{(2n+1)\pi\alpha}{2\beta}}\sin\frac{(2n+1)\pi}{2\beta}(\eta-\delta).$$

It may be observed that the results obtained for the case of three confocal parabolas can be obtained from those obtained in this article by putting $\xi_1=-\xi_2$ (so that $\gamma=0$) in this article and $\xi_1=a$ in §§ 2 and 3.

Another expression for ψ_1, with a corresponding expression for ϕ_1, is obtained by the assumption

$$\psi_1=\sum_1^\infty \sin\frac{n\pi}{2\alpha}(\xi-\xi_1)\left[A_n\sinh\frac{n\pi}{2\alpha}(\eta-\eta_1)+B_n\sinh\frac{n\pi}{2\alpha}(\eta_2-\eta)\right]$$

$$+\sum_1^\infty \sin\frac{n\pi}{2\beta}(\eta-\eta_1)\left[C_n\sinh\frac{n\pi}{2\beta}(\xi-\xi_1)+D_n\sinh\frac{n\pi}{2\beta}(\xi_2-\xi)\right].$$

In this case we shall find

$$A_n=\frac{8\omega\eta_2^2\left[(-1)^n\left(\frac{8a^2}{n^2\pi^2}-\xi_2^2\right)-\left(\frac{8a^2}{n^2\pi^2}-\xi_1^2\right)\right]}{n\pi\sinh\frac{n\pi\beta}{a}},$$

$$C_n=\frac{8\omega\xi_2^2\left[(-1)^n\left(\frac{8\beta^2}{n^2\pi^2}-\eta_2^2\right)-\left(\frac{8\beta^2}{n^2\pi^2}-\eta_1^2\right)\right]}{n\pi\sinh\frac{n\pi a}{\beta}},$$

B_n and D_n are what A_n and C_n become when η_1 is written for η_2 in A_n and ξ_1 for ξ_2 in C_n.

Also

$$\phi_1 = -\sum_1^\infty \cos\frac{n\pi}{2a}(\xi-\xi_1)\left[A_n\cosh\frac{n\pi}{2a}(\eta-\eta_1)+B_n\cosh\frac{n\pi}{2a}(\eta_2-\eta)\right]$$

$$+\sum_1^\infty \cos\frac{n\pi}{2\beta}(\eta-\eta_1)\left[C_n\cosh\frac{n\pi}{2\beta}(\xi-\xi_1)+D_n\cosh\frac{n\pi}{2\beta}(\xi_2-\xi)\right].$$

A third and simpler expression for ψ_1 can be obtained by assuming

$$\psi_1 = \sum_1^\infty \sin\frac{n\pi}{a}(\xi-\gamma)\left[A_n\sinh\frac{n\pi}{a}(\eta-\eta_1)+B_n\sinh\frac{n\pi}{a}(\eta_2-\eta)\right]$$

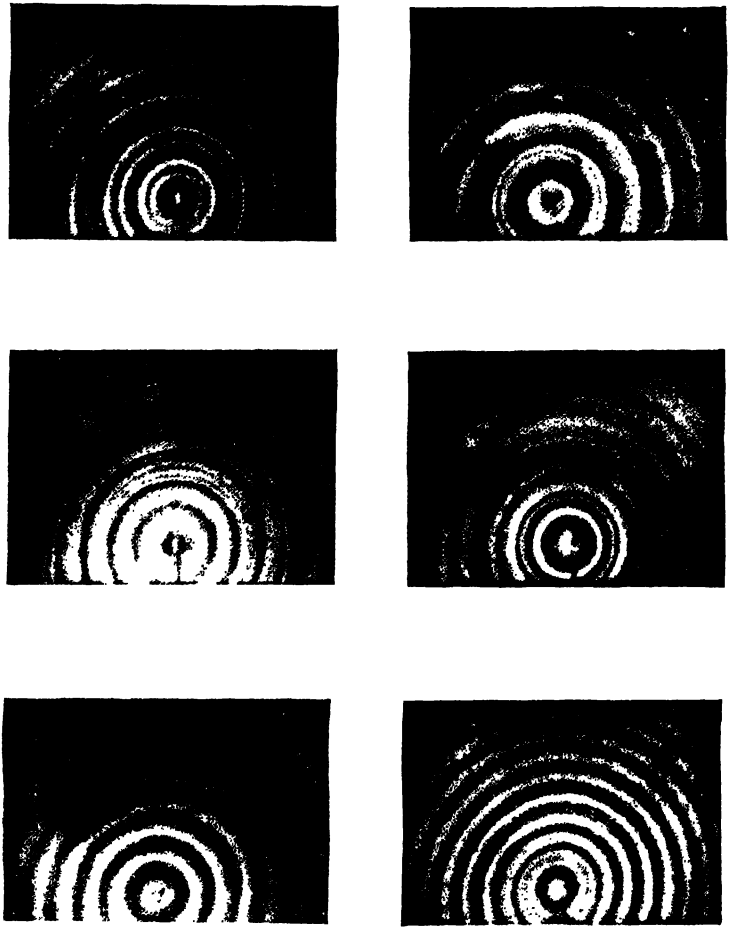
$$\sum_1^\infty \sin\frac{n\pi}{\beta}(\eta-\delta)\left[C_n\sinh\frac{n\pi}{\beta}(\xi-\xi_1)+D_n\sinh\frac{n\pi}{\beta}(\xi_2-\xi)\right].$$

It will be found that

$$A_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\eta_2^2\alpha\gamma}{\sinh\frac{2n\pi\beta}{\alpha}},\quad B_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\eta_1^2\alpha\gamma}{\sinh\frac{2n\pi\beta}{\alpha}},$$

$$C_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\xi_2^2\beta\delta}{\sinh\frac{2n\pi\alpha}{\beta}},\quad D_n = \frac{(-1)^{n+1}}{n\pi}\cdot\frac{16\omega\xi_1^2\beta\delta}{\sinh\frac{2n\pi\alpha}{\beta}}.$$

The corresponding expression for ϕ_1 can be easily deduced.



Ripples of Large Amplitude.

Investigations on Ripple Motion

By

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[Plate]

The problem of wave motion on the surface of liquids has for a long time been the subject of both theoretical and experimental study. Amongst the earlier workers were Scott Russel,[1] Cauchy[2] and Poisson[3]; the last two solved the problem of the effect of a local impulse on the surface of water. Soon after Stokes[4] gave a complete theory of surface waves of finite amplitude, without however taking the effect of surface tension into consideration, but Kelvin[5] gave the theory of capillary waves of small amplitude and proved the existance of a minimum velocity of propagation, which was afterwards experimentally verified by Rayleigh.[6] More recently Dr. Wilton[7] gave a theory of the form of ripples of finite amplitude and showed mathematically that the lengths of crests is greater than that of the hollows. While trying to verify Wilton's theory some remarkable results were obtained by the present author in the case of the larger wave lengths. It was found that for finite amplitudes the crests of waves begin to divide with increase of wave length, the form of the waves thus becoming complex with two or more crests in each wave length. It was also found that the form of the waves changes with the depth of liquid; for small depths the form becomes simpler, but the depth at which this change of form takes place is very small. As might be expected the depth at which the change of form occurs was found to be less for small lengths and greater for larger wave lengths.

[1] *Brit. Ass. Rep.*, Vol. VI, 1844.

[2] "Mémoire sur le théorie des ondes," *Mem. de l' Acad. Roy. des. Sciences*, t. 1. (1827).

[3] "Mémoire sur la théorie des oddes," *Mem.. de l' Acad. Roy. des Sciences*, t. 1. (1816), Lamb's Hydradynamics, 4th edition, p. 373).

[4] *Camb. Trans.*, t. VIII (1847) and *papers*, Vol. I, p. 197.

[5] *Baltimore Lectures*, p. 598; *Phil. Mag.* (4), t. XLVII, p. 374.

[6] *Phil. Mag.* (5), t. XXX, p. 386, *papers*, t. III, p. 394.

[7] *Phil. Mag.*, May, 1915, p. 688.

Ripples can be excited on the surface of a liquid either by periodic puffs of air directed against a point or by the periodic vertical motion of a dipper at a point on the surface of the liquid. The dipper can be attached to a tuning fork, but it was found more advantageous to attach it to Prof. Fleming's motor vibrator as improved by Prof. C. V. Raman.[1] By means of this instrument, we can vary the amplitude of the waves very conveniently and accurately, and also the wave length, by increasing or decreasing the speed of the motor by including a rheostat in its circuit. The ripples thus formed are seen by the method of stroboscopic observation. The waves thus seen, appear to be a series of concentric circles at equal distances in the case of small amplitudes and for large amplitudes and increasing wave lengths, the circles become double, triple and so on, corresponding to two or more crests for each wave length. Some photographs of those forms are shown in Figs. 1—6.

The phenomena observed above can be explained mathematically as follows. The simpler case of plane waves is taken up first.

Let the axis of x be in the direction of propagation of waves and the axis of y vertically upwards and the undisturbed level of the liquid as the plane $y=0$, and let the depth of the liquid be h. Then for irrotational two dimensional wave motion, the velocity potential satisfies the equation.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \qquad \ldots \quad (1)$$

$$\frac{\partial\phi}{\partial y} = 0, \quad \text{when } y=-h. \qquad \ldots \quad (2)$$

This condition denots that the velocity at the bottom is zero and the pressure condition at the surface gives,

$$-\frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = -gy + \frac{\partial\phi}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}, \qquad \ldots \quad (3)$$

taking the curvature to be small, which is true for pretty long waves. Since the disturbing force is simple harmonic, we can suppose the following expression for ϕ,

$$\phi = \sum_{s=1}^{s=n} a_s \cosh sm(y+h) \sin s\,m(ct-x). \qquad \ldots \quad (4)$$

which satisfies (1) and (2).

We have also the condition,

$$-\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial t}, \text{ and for } y=0,$$

[1] *Phys. Rev.*, Nov., 1919.

this gives

$$y=\frac{1}{c}\sum_{s=1}^{s=n} a_s \sinh smh \cos s\,m\,(ct-x) \qquad \ldots \quad (5)$$

To determine the constants a_1, a_2, a_3 etc. we substitute the values of ϕ and y in the pressure equation (3) and equate the co-efficients of $\cos sm(ct-x)$ to zero, we get approximate equations for the determination of c, a_2, a_3...etc, a_1 itself being arbitrary. For practical purposes it is sufficient to take only three terms. Thus we get the follcwing values for c, a_2 and a_3 :

$$c^2=\left(\frac{T m}{\rho}+g\right)\tanh mh-\frac{m^3}{2c}a_1^2\sinh^2 mh$$

$$-\tfrac{8}{3}m^2a_2\tanh mh\sinh^2 mh+\frac{m^2}{2}a_2\frac{\cosh 3mh}{\cosh mh}\,; \qquad \ldots \quad (6)$$

$$a_2=\left(\frac{m^2}{2}a_1^2\sinh^2 mh-\frac{m^2}{4}\right)$$

$$\Big/\left\{\left(\frac{4Tm^2}{\rho c}+\frac{g}{c}\right)\sinh 2mh-2mc\cosh 2mh\right.$$

$$-\frac{2m^2}{c}a_1^2\sinh^2 mh\cosh 2mh-\frac{m^3}{2c}a_1^2\sinh mh\sinh 2mh\cosh mh$$

$$\left.+\tfrac{3}{2}\frac{m^3}{c}a_1^2\sinh mh\sinh 3mh\right\},$$

and $a_3=\Big(\tfrac{3}{2}m^2a_1a_2\sinh mh\sinh 2mh-\tfrac{3}{2}\frac{m^3}{c}a_1a_2^2\sinh mh\sinh 3mh$

$$+\frac{m^3}{c}a_1a_2^2\sinh^2 2mh\Big)$$

$$\Big/\left\{\left(\frac{9Tm^2}{\rho c}+\frac{g}{c}\right)\sinh 3mh-3mc^3\cosh 3mh\right.$$

$$-c^2m^3a_1^2\left(\tfrac{1}{2}\sinh 3mh\sinh^2 mh+\frac{27}{4}\cosh 3mh\sinh^2 mh\right.$$

$$\left.\left.-\tfrac{3}{2}\sinh 4mh\sinh mh\right)\right\}.$$

Putting $\frac{a_1}{c}\sinh mh=a$, the formula for the wave form becomes

$$y=a\cos m\,(ct-x)$$

$$+\frac{\left(\frac{c^2m^2}{2}-\frac{m^2c^2}{4\sinh^2 mh}\right)\sinh 2mh\cos 2m\,(ct-x)}{\left(\frac{4Tm^2}{\rho}+g\right)\sinh 2mh-2mc^2\cosh 2mh-m^3a^2\left(2\cosh 2mh+\cosh^2 mh-\tfrac{3}{2}\frac{\cosh 3mh}{\cosh mh}\right)}.$$

$$+\left\{\tfrac{1}{2}cm^3aa_2\sinh 2mh+aa_2{}^2m^3\left(\frac{\sinh^2 2mh}{\sinh mh}-\tfrac{3}{2}\sinh 3mh\right)\right\}$$

$$\times\sinh 3mh\cos 3m\,(ct-x)$$

$$/\left\{\left(\frac{9\mp m^2}{\rho}+g\right)\sinh 3mh-3mc^2\cosh 3mh-c^2m^2a^2\left(\tfrac{1}{2}\sinh 3mh\right.\right.$$

$$\left.\left.+\frac{27}{4}\cos 3mh-\tfrac{3}{2}\frac{\sinh 4mh}{\sinh mh}\right)\right\}, \qquad \ldots\ (7)$$

where a_2 is the co-efficient in the second term of $\cos 2m\,(ct-x)$. This formula becomes for infinite depth,

$$y=a\cos m\,(ct-x)+\frac{\frac{m}{2}a^2c^2\cos 2m\,(ct-x)}{\left(\frac{4\mp m^2}{\rho}+g\right)-2mc^2+\frac{1}{2}m^3a^2c^2-m^3a'^2c^2}$$

$$+\frac{\frac{3}{2}c^2m^2aa'-\frac{1}{2}c^2m^3aa'^2}{\frac{9\mp m^2}{\rho}+g-3mc^2+\frac{3}{4}m^3c^2a^2+\frac{1}{4}m^3a'^2c^2}\cos 3m\,(ct-x), \qquad \ldots\ (8)$$

where

$$\frac{4\mp m^2}{\rho}+g-2mc^2+\tfrac{1}{2}m^3a^2c^2$$

This formula fully explains the phenomenon observed. For large gravity waves the formula reduces to

$$y=a\cos m(ct-x)-\tfrac{1}{2}ma^2\cos 2m\,(ct-x)+\tfrac{3}{8}m^2a^3\cos 3m\,(ct-x).$$

This shows that the form of the waves becomes cycloidal as was pointed out by Stokes.

To explain the change of form of waves with increasing distance from the source, we use cylindrical coordinates. We take the axis of z vertically upwards and as before the undisturbed surface of the water as the plane $z=0$.

The equation are modified as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0. \qquad \ldots\ (9)$$

$$\frac{\partial\phi}{\partial r}=0, \text{ when } r=\infty,$$

$$\frac{\partial\phi}{\partial z}=0. \text{ when } z=-h. \qquad \ldots\ (10)$$

The pressure equation becomes

$$-\frac{T}{\rho}\frac{\partial^2\phi}{\partial r^2}=-gz+\frac{\partial\phi}{\partial t}-\tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2\right\}. \quad \ldots (11)$$

We can therefore assume

$$\phi=\sum_{s=1}^{s=n} a_s J_s(skr)\cosh sk(z+h)\sin skct, \quad \ldots (12)$$

and from the condition $-\frac{\partial\phi}{\partial z}=\frac{\partial z}{\partial t}$, we get for $z=0$,

$$z=\frac{1}{c}\sum_{s=1}^{s=n} a_s J_s(skr)\sinh skh\cos skct. \quad \ldots (13)$$

Substituting these values in the pressure equation (11) and taking only three terms into consideration as in the case of plane waves, and equating the co-efficients of $\cos sm(ct-x)$ to zero, we get equations for the determination of c, a_2 and a_3, a_1 itself being arbitrary.

Thus we get

$$c^2=-\frac{Tk}{\rho}\frac{J_0''(kr)}{J_0(kr)}\tanh kh+\frac{g}{k}\tanh kh-\frac{k}{2}a_1^2\{J_0(kr)\}^2$$

$$+k^2a_2\frac{J_1(kr)}{J_0(kr)}.J_1(2kr)\cosh 2kh-\frac{3}{2}\frac{k^2}{c}a_2J_0(2kr)\tanh kh\sinh 2kh,$$

since $$J_0''(kr)=-J_0(kr)+\frac{1}{kr}J_1(kr),$$

we have as a first approximation

$$c^2=\left(-\frac{Tk}{\rho}\frac{J_0''(kr)}{J_0(kr)}+\frac{g}{k}\right)\tanh kh$$

$$=\left(\frac{Tk}{\rho}-\frac{Tk}{\rho}\frac{1}{kr}\frac{J_1(kr)}{J_0(kr)}+\frac{g}{k}\right)\tanh kh$$

and

$$z=\frac{a_1}{c}\sinh kh\,J_0(kr)\cos kct$$

$$+\frac{1}{c}\left(\left[\tfrac{3}{2}k^2a_1^2\{J_0(kr)\}^2\sinh^2 kh+\frac{k^2a_1^2}{4}\{J_1(kr)\}^2\cosh^2 kh\right]\right.$$

$$\sinh 2kh\,J_0(2kr)\cos 2kct\Big)\Big/\left(-\frac{1}{4}\frac{Tk^2}{\rho}J_0''(2kr)\sinh 2kh\right.$$

$$\left.+\frac{g}{c}J_0(2kr)\sinh 2kh-2kcJ_0(2kr)\cosh 2kh\right)$$

$$+\left\{\frac{1}{c}\left(\tfrac{3}{2}a_1a_2J_0(2kr)\sinh kh\sinh 2kh-a_1a_2J_1(2kh)\cosh kh\cosh 2kh\right)\right\}$$

$$\times\sinh 3kh\,J_0(3kr)\cos 3kct$$

$$\int\left(\frac{9\mp k^2}{\rho c}\ J_0''(3kr)\sinh 3kh+\frac{g}{c}\ J_0(3kr)\sinh 3kh\right.$$
$$\left.-3kc\ J_0(3kr)\cosh 3kh-\tfrac{3}{2}\ a_1a_2,\quad \ldots\quad (14)\right.$$

when a_2 is the co-efficient of the 2nd term.

Putting $\frac{a_1}{c}\sinh kh=a$ and simplifying we get,

$$z=a\ J_0(kr)\cos kct$$
$$+\frac{\left[\tfrac{3}{4}c^2k^2a^2\{J_0(kr)\}^2+\frac{k^2c^2a^2}{4}\ J_1(kr)\}^2\ \cosh^2 kh\right]\sinh 2kh\cos 2kct}{-\frac{4\mp k^2}{\rho}\ \frac{J_0''(2kr)}{J_0(2kr)}\sinh 2kh+g\sinh 2kh-2kc^2\cosh 2kh}$$
$$+\text{etc.}$$

For infinite depth, this becomes

$$z=a\ J_0(kr)\cos kct$$
$$+\frac{\tfrac{3}{4}c^2k^2a^2\{J_0(kr)\}^2+\frac{c^2k^2a^2}{4}\ \{J_1(kr)\}^2}{\left(-\frac{4\mp k^2}{\rho}\ \frac{J_0''(2kr)}{J_0(2kr)}+g\right)-2kc^2+\ldots}\cos 2kct$$
$$+\text{etc.}$$

The asymptotic value of $J_0(kr)$ is

$$\sqrt{\frac{2}{\pi kr}}\cos\left(kr-\frac{\pi}{4}\right),$$

and that of $J_1(kr)$ is $\sqrt{\frac{2}{\pi kr}}\sin\left(kr-\frac{\pi}{4}\right)$.

Substituting these values in the above formula, we get

$$z=a\sqrt{\frac{2}{\pi kr}}\cos\left(kr-\frac{\pi}{4}\right)\cos kct$$
$$+\frac{2}{\pi kr}\ \frac{\frac{1}{4}k^2a^2(1+2\sin 2kr)\cos 2kct}{-\frac{4\mp k^2}{\rho}\ \frac{J_0''(2kr)}{J_0(2kr)}+g-2kc^2}+\text{etc.}$$

The above formula shows that the amplitude of the second harmonic decreases inversely as the distance from the origin, whereas the fundamental decreases inversely as the square root of the distance. Hence the wave form would tend to become simpler with the increase of distance from the source and this is what is also found experimentally.

Note on the product of any number of Legendre functions of different degrees

By

Sudhansukumar Banerji

In 1878, Adams published in the Proceedings of the Royal Society of London, the well known result on the product of two Legendre functions of different degrees, namely

$$P_m(z)\,P_n(z)=\sum_{r=0}^{m}\frac{2n+2m-4r+1}{2n+2m-2r+1}\cdot\frac{A(n-r)A(m-r)A(r)}{A(n+m-r)}P_{n+m-2r}(z)$$

where
$$A(m)=\frac{1.3.5\ldots(2m-1)}{m\,!}, \qquad \ldots\ (1)$$

m and n being positive integers and $m\leq n$.

This result can also be written in the form

$$P_m(z)\,P_n(z)=\sum_{q=n-m}^{n+m}\frac{2q+1}{m+n+q+1}\cdot\frac{A\left(\frac{n+m-q}{2}\right)A\left(\frac{n+q-m}{2}\right)A\left(\frac{m+q-n}{2}\right)}{A\left(\frac{n+m+q}{2}\right)}P_q(z)$$

$$=\sum_{q=n-m}^{n+m}A_{n,m,q}\,P_q(z), \qquad \ldots\ (2)$$

where

$$A_{n,m,q}=\frac{2q+1}{m+n+q+1}\cdot\frac{A\left(\frac{n+m-q}{2}\right)A\left(\frac{n+q-m}{2}\right)A\left(\frac{m+q-n}{2}\right)}{A\left(\frac{n+m+q}{2}\right)}$$

and the summation extends for the following values of q

$$n-m,\ n-m+2,\ \ldots n+m.$$

From this Adams deduced the value of the integral

$$\int_{-1}^{1} P_m(z)\, P_n(z)\, P_q(z)\, dz$$

$$= \frac{2}{m+n+q+1} \frac{A\left(\frac{n+m-q}{2}\right) A\left(\frac{m+q-n}{2}\right) A\left(\frac{q+n-m}{2}\right)}{A\left(\frac{n+m+q}{3}\right)} \quad \ldots (3)$$

and q takes the values

$$n-m,\quad n-m+2,\quad n-m+4,\ldots n+m,\quad \text{if } n>m$$

or

$$m-n,\quad m-n+2,\quad m-n+4,\ldots m+n,\quad \text{if } m>n$$

and for other values of q the integral is zero.

The method can easily be extended to obtain an expression in a series of Legendre functions for the product of three or more Legendre functions and when this has been done the values of similar integrals involving four or more Legendre functions can be at once written down. It is interesting to note the results in the different cases and the conditions under which they hold.

If (2) be multiplied by $P_s(z)$, where s is a positive integer and if $s>n+m$, we have

$$P_m(z)\, P_n(z)\, P_s(z)$$

$$= \sum_{q=n-m}^{n+m} P_q(z)\, P_s(z)$$

$$= \sum_{q=n-m}^{n+m} A_{n,m,q} \sum_{t=s-q}^{s+q} A_{s,q,t}\, P_t(z), \quad \ldots (4)$$

$$= A_{n,m,n+m}\, A_{s,n+m,s-n-m}\, P_{s-n-m}(z)$$

$$+ (A_{n,m,n+m-2}\, A_{s,n+m-2,s+2-n-m} + A_{n,m,n+m}\, A_{s,n+m,s+2-n-m})\, P_{s+2-n-m}(z)$$

$$+ (A_{n,m,n+m-4}\, A_{s,n+m-4,s+4-n-m} + A_{n,m,n+m-2}\, A_{s,n+m-2,s+4-n-m} + A_{n,m,n+m}\, A_{s,n+m,s+4-n-m})\, P_{s+4-n-m}(z)$$

$$+ \ldots + (A_{n,m,n+m}\, A_{s,n+m,s+n+m})\, P_{s+n+m}(z). \quad \ldots (5)$$

From this the law of formation of the co-efficients is obvious.

If $s \leq n-m$, we have

$$P_m(z)\, P_n(z)\, P_s(z) = \sum_{q=n-m}^{n+m} A_{n,m,q} \sum_{t=q-s}^{q+s} A_{q,s,t}\, P_t(z)$$

$$= A_{n,m,n-m}\, A_{n-m,s,n-m-s}\, P_{n-m-s}(z)$$

$$+ (A_{n,m,n-m+2}\, A_{n-m+2,s,n-m-s+2} + A_{n,m,n-m}\, A_{n-m,s,n-m-s+2})\, P_{n-m-s+2}(z)$$

$$+ (A_{n,m,n-m+4}\, A_{n-m+4,s,n-m-s+4} + A_{n,m,n-m+2}\, A_{n-m+2,s,n-m-s+4}$$

$$+ A_{n,m,n-m}\, A_{n-m,s,n-m-s+4})\, P_{n-m-s+4}(z)$$

$$+ \ldots + A_{n,m,n+m}\, A_{n+m,s,n+m+s}\, P_{n+m+s}(z).$$

If $n-m<s<n+m$, let $s=n-m+r$, where r may be an odd or an even integer $(0<r<2m)$.

If r is an odd integer, then

$$P_n(z)\, P_m(z)\, P_s(z)$$

$$= \sum_{q=n-m}^{n-m+r-1} A_{n,m,q} \sum_{t=n-m+r-q}^{n-m+r+q} A_{n,\, m+r,q,t}\, P_t(z)$$

$$+ \sum_{q=n-m+r+1}^{n+m} A_{n,m,q} \sum_{t=q-n+m-r}^{q+n-m+r} A_{q,n-m+r,t}\, P_t(z).$$

If r is an even integer, then

$$P_n(z)\, P_m(z)\, P_s(z)$$

$$= \sum_{q=n-m}^{n-m+r} A_{n,m,q} \sum_{t=n-m+r-q}^{n-m+r+q} A_{n-m+r,q,t}\, P_t(z)$$

$$+ \sum_{q=n-m+r+2}^{n+m} A_{n,m,q} \sum_{t=q-n+m-r}^{q+n-m+r} A_{q,n-m+r,t}\, P_t(z).$$

These cover all the cases that may arise for the product of three Legendre functions.

To evaluate the integral

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_s(z)\, P_t(z)\, dz,$$

it should be noticed that if $s>n+m$, then the integral will have a definite value other than zero if t takes any of the values

$$s-n-m,\quad s-n-m+2,\quad s-n-m+4, \ldots s+n+m.$$

and for other values of t, the integral is zero.

When t has any of the values given above the value of the integral can be written in the form

$$\frac{2}{2t+1}\sum_{r}A_{n,m,n-m+2r}\sum_{r'}A_{s,n-m+2r,\,s-n+m+2r}(r-r')$$

where the first summation extends for all integral values of r form 0 to m and the second summation for all integral values of r' from 0 to $n-m+2r$, which make

$$r-r'=\text{a given integer}=\frac{(s+m)-(n+t)}{2}\,.$$

The following particular cases are interesting :

$$\int_{-1}^{1}P_s(z)\,P_m(z)\,P_n(z)\,P_{s-n-m}(z)dz$$

$$=\frac{2}{2s-2n-2m+1}\,A_{s,m,s+m}\,A_{n,s+m,s-n-m}$$

$$\int_{-1}^{1}P_s'(z)\,P_m(z)\,P_n(z)\,P_{s-n\ m+2}(z)dz$$

$$=\frac{2}{2s-2n-2m+5}\,\{A_{s,m,s+m-2}\,A_{n,s+m-2,s+2-n-m}$$

$$+A_{s,m,s+m}\,A_{n,s+m,s+2-n-m}],$$

$$\int_{-1}^{1}P_n(z)P_m(z)P_s(z)P_{s-n-m+4}(z)\,dz$$

$$=\frac{2}{2s-2n-2m+9}\,[A_{s,m,s+m-4}\,A_{n,s+m-4,s+4-n-m}$$

$$+A_{s,m,s+m-2}\,A_{n,s+m-2,s+4-n-m}+A_{s,m,s+m}\,A_{n,s+m,s+4-n-m}],$$

etc.

When $s<n-m$, the integral

$$\int_{-1}^{1}P_n(z)\,P_m(z)\,P_s(z)\,P_t(z)dz$$

has a value other than zero when t takes any of the values

$$n-m-s,\quad n-m-s+2,\quad n-m-s+4,\ldots n+m+s,$$

and for other values of t, the integral is zero.

The value of the integral when t has any of the above values can be easily written down. The following particular cases are interesting :

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_s(z)\, P_{n-m-s}\, dz$$

$$= \frac{2}{2n-2m-2s+1} \cdot A_{n,m,n-m}\, A_{n-m,s,n-m-s},$$

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_s(z)\, P_{n-m-s+2}(z)\, dz$$

$$= \frac{2}{2n-2m-2s+5} \left[A_{n,m,n-m+2}\, A_{n-m+2,s,n-m-s+2} + A_{n,m,n-m}\, A_{n-m,s,n-m-s+2} \right],$$

etc.

If $n-m<s<n+m$, let $s=n-m+r$, where r may be an odd or even integer $(0<r<2m)$. If r is an odd integer, then the integral

$$\int_{-1}^{1} P_n(z)\, P_m(z)\, P_r(z)\, P_t(z)\, dz$$

will have a value other than zero, if t takes any of the values

$$1, 3, 5, \ldots 2n-2m+2r-1, \ldots 2n+r,$$

and for other values of t, the integral is zero.

If however r is an even integer, then the integral has a value other than zero if t takes any of the values

$$0, 2, 4, \ldots 2n-2m+2r, \ldots 2n+r.$$

and for other values of t the integral is zero.

The value of the integral when t has any of the values enumerated above can be easily written down.

The method can obviously be extended to the product of four and more Legendre functions of different degrees but the conditions under which the different expressions hold become more and more numerous and the results more and more complicated.

As an application of the integral of the product of three Legendre functions of different degrees, it may be mentioned here that it was shown by the author in a previous paper published in this Bulletin that if (r, θ) and (r', θ') represent the polar co-ordinates of a point measured from two given poles, then

$$\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}} P_n(\mu) = \sum_{p=0}^{\infty} C_{n,p} \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu'),$$

where $\cos\theta = \mu$, $\cos\theta' = \mu'$ and

$$C_{n,p} = (2p+1) i^{p+n} \int_{-1}^{1} P_p(z) P_n(\cos\theta) e^{-ik} P_a$$

$$= (2p+1) i^{p+n} \sqrt{\frac{\pi}{2kD}} \sum_{s=0}^{\infty} i^{s} (2s+1) J_{s+\frac{1}{2}}(kD)$$

$$\int_{1}^{1} P_p(z) P_n(z) P_s(z) dz$$

$$= (2p+1) i^{p+n} \sqrt{\frac{\pi}{2kD}} \sum_{s} i^{s} (2s+1) J_{s+\frac{1}{2}}(kD) A_{p,n,s} \frac{2}{2s+1},$$

D being the distance between the poles and the summation with respect to s extends from $p-n$ to $p+n$ or $n-p$ to $p+n$.

For this and similar results reference is made to two papers published by the author in the Bulletin of the Society, Vol. IV and Vol. V.

Other applications of the results obtained in this paper will be published in due course.

PHYSICS

Ionisation in the Solar Chromosphere

BY

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It has been known for a long time that the high level chromosphere is generally distinguished by those lines which are relatively more strengthened in the spark than in the arc, and which Lockyer originally styled as enhanced lines. The following list taken from Mitchell's[1] list of chromospheric lines along with the respective intensities of the line in the arc, and the spark, illustrates the case.

Table I.

Element.	Wave-length of the line in A. U's.	Chromospheric level reached in Kms.	Intensity.[2] Arc.	Spark.
Calcium ...	H—3968 ...	14000	500	1000
	K—3933 ...	14000	300	500
	g—4226 ...	5000	1000	L100
Strontium ...	4215·88	6000	500	L500
	4077·98	6000	1000	L1000
Barium ...	4994 ...	750	500	L500
	4554 ...	1200	1000	L1000
Scandium ...	4247 ...	6000	50	1000
Titanium ...	3685·41	6000	8	100
	3741·78	1500	3	10
	3759·47	6000	10	L20
	3761·47	6000	6	L10
	3900·68	1600	5	L50
	3913·60	2000	5	L20
	4290·33	1300	2	L10

[1] Mitchell—The Astrophysical Journal, Vol. 38, p. 424.

[2] L denotes "enhanced" according to Lockyer.

Element.	Wave-length of the line in A. U's.	Chromospheric level reached in Kms.	Intensity.[1] Arc.	Spark.
Titanium	4294·23	1200	2	L10
	4300·31	1200	3	L8
	4395·29	2500	10	L20
	4444·01	1600	4	L15
	4468·71	1500	4	L15
	4501·75	1600	4	L15
	4563·93	1200	3	L10
	4572·17	1200	5	L20

This table makes it clear that generally only those lines occur in the highest levels which are relatively more strengthened in the spark discharge spectrum. We can add to the above table a list of the high-level iron, and other lines, but the conclusion would be the same.

It appears that no satisfactory explanation of this fact, as well as of the extraordinary height reached by these lines has yet been offered. It is intimately connected with the physical mechanism of the arc and the spark. In this connection it is well to recall Lockyer's original hypothesis, which, however does not seem to have been, at any time, much in favour with the physicists. According to Lockyer, the passage from the arc to the spark means a great, though localised, increase of temperature, to which mainly the enhancement of the lines were to be ascribed. But apart from its physical incompleteness, Lockyer's theory launches us amidst great difficulties as far as the interpretation of the solar phenomena is concerned. It would lead us to the hypothesis that the outer chromosphere is at a substantially higher temperature than the photosphere, and the lower chromosphere, in other words, the temperature of the sun increases as we pass radially outwards. This hypothesis is however, quite untenable and is in flagrant contradiction to all accepted theories of Physics.

A much more plausible explanation is that the lines in question are not due to radiations from the normal atom of the element, but from "an ionised atom, *i.e.*, one which has lost one electron." The high-level chromosphere is, according to this view, therefore the seat

[1] L denotes "enhanced" according to Lockyer.

of very intense ionisation. Let us see briefly how this hypothesis has grown up.

Modern theories of atomic structure and radiation leave little doubt that the "Enhanced lines" are due to the ionised atom of the element. As a concrete example, let us take the case of the Calcium H, k and g lines. The 'H-k' lines are of the enhanced type, while g is of the normal type. The 'H-k' are the leading members of the principal pair-series of the system of double lines of Calcium, while the g-line is the first member of the system of single lines of Calcium. Lorenser, and Fowler[1] have shown that the series formula for the double lines is of the type

$$v = 4N \left[\frac{1}{\{f(m)\}^2} - \frac{1}{\{\phi(n)\}^2} \right]$$

while the series formula for the single lines is of the type,

$$v = N \left[\frac{1}{\{f'(m)\}^2} - \frac{1}{\{\phi'(n)\}^2} \right]$$

where $f(m)$, $\phi(n)$ are functions of the form $m+\alpha$ or

$m+\alpha+\beta[t(m)]$, when $t(m)$ diminishes with increasing values of m.

In other words, in the series formula of the enhanced lines, the spectroscopic constant is 4N instead of the usual Rydberg number N. In the light of Bohr's theory, this is to be understood in the sense that during the emission of the enhanced lines, the nucleus, and the system of electrons (excluding the vibrating one) taken together behave approximately as a double charge, so that the spectroscopic constant, $N = \frac{2\pi^2 e^2 E^2 m}{h^3}$ becomes 4N, as $E = 2e$. This means that if the nuclear charge is n, the total number of electrons is $(n-1)$, and the system has been produced by the removal of one electron from the normal atom.

What has been said of the Calcium lines H and K, is also true of the Strontium pair, 4215 and 4077, and the Barium-pair 4934, and 4554, *i.e.*, they are due to the ionised atom of these elements.

[1] Fowler—Phil. Trans., Vol. 214.

The principal lines of the system of single lines of these elements also occur in the flash spectrum, but the following table shows that they reach a much lower level.

TABLE II.

Element.	Lines due to the Ionised atom.	Chromospheric Level.	Lines due to the Normal atom.	Chromospheric Level.
Ca.	H−3968 ... K−3933 ...	14000	g-4227	5000
Sr.	4215 ... 4077 ...	6000	4607	350
Ba.	4934 ... 4554 ...	750 1200	5536	400

No satisfactory series formula are known for the other high-level chromospheric elements, *viz.*, Titanium, Scandium, Iron, and other elements. But the recent remarkable work of Kossel and Sommerfeld[1] makes it quite clear that the spark lines of these elements are due to the ionised atom. The spark lines of alkalies have not been much investigated and lie in the ultraviolet beyond 3,000 so that even if they are present in the high[2] level atmosphere, we shall have no means of detecting them.

As regards hydrogen, ionised hydrogen would mean simply the hydrogen core, and this probably by itself would be incapable of emitting any radiation. But as H_α and H_β lines occur high in the chromosphere, we have to admit that Hydrogen probably is not much ionised in the chromosphere.

The case of Helium is very interesting. It is well-known that the Fraunhofer spectrum does not contain any helium lines, which are obtained only in the flash spectrum. But these lines are all due to normal helium, and the well-known D_3 reaches a level of 7500 kms. The lines due to ionised helium are represented by the general series formula

$$\nu = 4N\left\{\frac{1}{m^2} - \frac{1}{n^2}\right\}.$$

[1] Kossel and Sommerfeld—Ber. *d.* D. Phys. Gesselschaft.

[2] Kossel and Sommerfeld—loc. cit., p. 250, Jahrgang. 21, p. 240.

and the best known of them in the visible range, are the Rydberg line 4686, and the Pickering system $\nu = N\left[\frac{1}{2^2} - \frac{1}{(m+\frac{1}{2})^2}\right]$ once ascribed to "cosmic hydrogen." Mitchell[1] states that 4686 occurs in the flash spectrum, and reaches a level of 2000 kms. If the identification be all right, helium would present a seemingly anomalous case, for whereas other elements are ionised in the upper strata, it is ionised in the lower strata of the chromosphere.

The above sketch embodies, in short, the problems before us. The alkaline earths, and the heavier elements are ionised through the whole of the solar atmosphere, but the ionisation is complete in the high-level chromosphere, which seems to contain no normal atom at all. But hydrogen and helium are probably unionised throughout the whole chromosphere, and in the case of helium, we have probably some slight ionisation in the lower parts,—a rather anomalous case.

The explanation of these problems and some other associated problems of solar physics, will be attempted in this paper. The method is based upon a recent work of Eggert[2] "on the state of dissociation in the inside of fixed stars." In this paper, Eggert has shown that by applying Nernst's formula of "Reaction-isochore" to the problems of gaseous equilibrium in the inside of the stars, it is possible to substantiate many of the assumptions made by Eddington[3] in his theory of the constitution of the stars.

These assumptions are that in the inside of the stars, the temperature is of the range of $10^6 - 10^7$ degrees, and the pressure is about 10^6 Atm, and the atoms are so highly ionised that the mean atomic weight is not much greater than 2. The method of Eggert is directly applicable in the study of the problems sketched above. The equation of the Reaction-isochore is

$$\log k_p = -\frac{U}{4\cdot571T} + \frac{\Sigma\nu C_p}{R} + \Sigma\nu c.$$

where
K = the Reaction-isochore,
U = heat of dissociation,
C_p = Specific heat at constant pressure,
c = Nernst's chemical constant,

[1] Mitchell—loc. cit., pp. 490-491.

[2] Eggert.—Phys. Zeitschrift, Dec. 1919.

[3] Eddington—Month. Not. Roy. As. Society, Vol. 77, pp. 16 and 596.

and the integration is extended over all the reacting substances. The present case is treated as a sort of chemical reaction, in which we have to substitute ionisation for chemical decomposition. The next section shows how U is to be calculated. The equation will be resumed in §3.

§2. We may regard that the ionisation of a calcium atom takes place according to the following scheme familiar in physical chemistry,

$$Ca \rightleftarrows Ca^+ + e - U$$

where Ca is the normal atom of calcium (in the state of vapour). Ca^+ is an atom which has lost one electron, U is the quantity of energy liberated in the process. The quantity considered is 1 gm. atom.

The value of U in the case of alkaline earths, and many other elements can easily be calculated from the value of the ionisation potential of elements as determined by Frank and Hertz, Mac-Lennan[1] and others. Let V = ionisation potential. Then to detach one electron from the atomic system, we must add to each atom an amount of energy equivalent to that acquired by an electron falling through a potential difference V, where V (in volts) is given by the quantum relation,

$$\frac{e\,V}{300} = h\,\nu_0$$

ν_0 being the convergence frequency of the principal series, *i.e.*, $(1, s)$ in Paschan's notation. If this quantity be multiplied by the Avogadro number N, and expressed in Calories, we obtain U.

Thus if V = 1 volt, we have

$$U = \frac{e\,V\,N}{J.\,300} = \frac{9645 \times 10^8}{4{\cdot}2 \times 10^7} = 2{\cdot}302 \times 10^4 \text{ calories.}$$

(Using $Ne = 9645C$, C = velocity of light.)

Table 3 contains, for future use, the values of the ionisation potentials as far as known, and the calculated value of U.

Here I wish to remark that an element may have more than one ionisation potential, depending upon the successive transfer of the

[1] Mac-Lennan—Proceedings of the Physical Society, London, Dec. 1918.

outer electrons one by one to infinity, or the simultaneous existence of two different constitutions of the normal atom (*e.g.*, Helium, and parhelium). The ionisation potentials given below correspond to the case when only one electron is transferred to infinity leaving an excess of unit positive charge in the atom. We have made it clear in the introduction that the high level alkaline earth-lines are due to the atoms with one plus charge in excess.

TABLE III.

Element.	Ionisation Potential.	U in Calories.	
Mg	7·65	1·761	10^5 Calories.
Ca	6·12	1·409	,,
Sr	5·70	1·312	,,
Ba	5·21	1·199	,,
Ra			
—			
Na	5·112	1·177	,,
K	4·318	·994	,,
Rb	4·155	·956	
Cs	3·875	·891	
—			
Zn	9·4	2·164	
Cd	·9	2·071	
Hg	10·45	2·405	

The cases of Hydrogen and Helium will be taken later on.

Equation of the Reaction-isochore for Ionisation.

As mentioned in the introduction, gaseous equilibrium proceeds according to the equation.

$$\log K = -\frac{U}{4{\cdot}571\,T} + \frac{\Sigma\nu C_p}{R}\log T + \Sigma\nu C \qquad \ldots \quad (1)$$

when the reaction proceeds according to the scheme

$$\nu_a A + \nu_b B + \cdots = \nu_m M + \nu_n N + \ldots$$

and K is the "Reaction-isochore"

$$\frac{P_M^{\nu_m} P_N^{\nu_n}}{P_A^{\nu_a} P_B^{\nu_b}}$$

P_M, P_N, ... being the partial pressures of the reacting substances.

In the present cases, *viz.*, for a reaction of the type,

$$C_a = C_a^+ + e - U.$$

we have $$\Sigma\nu C_P = \left(C_P\right)_{C_a^+} + \left(C_P\right)_e - \left(C_P\right)_{C_a}$$

we can take $$\left(C_P\right)_{C_a^+} = \left(C_P\right)_{C_a},$$

and $\left(C_P\right)_e = \frac{5}{2} R$, the electron being supposed to behave like a monatomic gas.

Eggert calculates the chemical constant from the Sackur-Tetrode-Stern relation

$$C = -1{\cdot}6 + \tfrac{3}{2} \log M \text{(correct Rel}^n C = \log \frac{(2\pi m)^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3}$$

where M = molecular weight. P's are expressed in atmospheres.

Now C has the same value for C_a and C_a^+. For the electron $M = 5{\cdot}5 \times 10^{-5}$ we have thus $\Sigma\nu c = -6{\cdot}5$.

To calculate the 'Reaction-isobar' K, let us assume that P is the total pressure, and a fraction x of the Ca-atoms is ionised.
Thus we obtain

$$K = \left(\frac{x}{1+x}\right)^2 P^2 \left(\frac{1-x}{1+x}\right) P = \frac{x^2}{1-x^2} P.$$

and

$$\log \frac{x^2}{1-x^2} P = -\frac{U}{4{\cdot}571\,T} + 2{\cdot}5 \log T - 6{\cdot}5 \qquad \ldots \quad (2)$$

This is the equation of the Reaction-isochore which enables us to calculate the affinity of the ionised atom for the electron.

Ionisation of Calcium, Strontium and Barium.

With the aid of formula (2) the degree of ionisation for any element under any temperature and pressure can be calculated when the ionisation potential is known. As a concrete example, we may begin with Calcium, Strontium and Barium.

A glance at equation (2) shows that pressure has a very great influence on the degree of ionisation, which does not seem to have been anticipated. This is due to the occurrence of P in the first power in the expression for the 'Reaction-isochore' (*vide* equation 2). A reduction in the value of P is attended with greatly enhanced ionisation. This will become apparent from an inspection of the following table which show the ionisation of Calcium, Strontium, and Barium under varying conditions of pressure and temperature.

TABLE IV.

Ionisation of Calcium (in per cents).

U = volts = $1 \cdot 400 \times 10^3$ calories (approximately).

Pressure in atmospheres-Temperature on the absolute scale.

Pressure	10		10^{-}	10^{-2}	10^{-}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
Temp.									
2000°	...	$1 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	10^{-4}	$3 \cdot 10^{-4}$	10^{-3}	10^{-2}	10^{-1}	1
3000°	...	$7 \cdot 10^{-3}$	$2 \cdot 10^{-2}$	$7 \cdot 10^{-2}$	$2 \cdot 10^{-1}$	$7 \cdot 10^{-1}$	7	59	
4000°	...	$2 \cdot 10^{-1}$	$7 \cdot 10^{-1}$	2	6	19	56	99	
5000°	·53	1·7	5·4	17	47	84	99		
6000°	2·2	7	23	57	91	98			
7000°	6	20	53	90	99				
7500°	9	30	70	95					
8000°	15	42	83	97·5			Complete Ionisation.		
9000°	29	64	93	99					
10000°	42	82	97						
11000°	60	92	99						
12000°	75	96							
13000°	84	97							
14000°	90	98							
15000°	92								
16000°	96								

TABLE V.

Ionisation of Strontium.

U = 5·7 volts = $1 \cdot 30 \times 10^5$ calories (approximately).

Pressure	10	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-6}	10^{-8}
Temp.								
2000	...	...	...	...	$1 \cdot 10^{-3}$	$3 \cdot 5 \cdot 10^{-3}$	...	...
2500	...	...	...	...	$5 \cdot 10^{-1}$	$1 \cdot 4 \cdot 10^{-1}$	...	...
3000	...	4×10^{-2}	$1 \cdot 6 \times 10^{-1}$	$4 \cdot 10^{-1}$	1·6	5	46	85
4000	...	·4	1·3	3·8	12	36	96·5	
5000	·9	3	9	27	66	93		
6000	6	20	33	73	96			
7000	9	28	67	93				
7500	13	40	80	97·5				
8000	19	53	89	98·5				
9000	35	75	96·5			Complete Ionisation.		
10000	52	88	98·5					
11000	68	94						
12000	80	97·5						
13000	87	98·5						
14000	92							
15000								

TABLE VI.

Ionisation of Barium (in per cents).

U = 5·12 volts = 1·20 × 105 calories (approximately).

Pressure	10	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
Temp.								
2000	...	$2\cdot10^{-4}$	$6\cdot10^{-4}$	$2\cdot10^{-3}$	$6\cdot10^{-3}$	$2\cdot10^{-2}$	$2\cdot10^{-1}$	18
2500	...	$7\cdot10^{-3}$	$1\cdot8\cdot10^{-2}$	$7\cdot10^{-2}$	$1\cdot8\cdot10^{-1}$	$7\cdot10^{-1}$	6	48
3000	$1\cdot6\cdot10^{-2}$	$5\cdot2\cdot10^{-2}$	$16\cdot10^{-2}$	$52\cdot10^{-2}$	1·6	5	46	85
4000	$3\cdot10^{-1}$	$9\cdot10^{-1}$	3	9	28	68	99	
5000	1·7	5·5	17	48	86	98		
6000	6·2	19	52	88	99			
7000	15	43	83	98				
7500	22	57	91			Complete Ionisation.		
8000	30	70	94					
9000	...	85						
10000	65	93						
11000	79	97						
12000								

We are not aware how the temperature and the pressure (partial pressure for a particular element) varies with height in the solar atmosphere. According to F. Biscoe,[1] the temperature of the photosphere is about 7500K, while the pressure in the reversing layer varies, according to different investigators, from 10 to 1 atmospheres. If we suppose that the variation in temperature is entirely caused by radiation, the temperature of the upper layers should tend to the limit $\frac{7500}{2\frac{1}{4}}$,[2] or a little more than 6000K. The partial pressure may be supposed to vary from 10 atmospheres in the reversing layer to 10^{-10} atmospheres in the outermost layers.

An examination of tables (4—6) shows that, under the above-mentioned assumptions, about 34% of the *Ca*-atoms are ionised on the photosphere, when the pressure falls to 10^{-} atmospheres, almost all the atoms get ionised, so that up to this point in the solar atmosphere, we shall get combined emission of the H, K, and *g*-lines but above this point, we shall have only the H, K lines. This is in very good agreement with observed facts.

In the case of Strontium, and Barium, owing to their comparatively low ionisation potential, ionisation at 6000′ is practically

[1] F. Biscoe—The Astrophysical Journal, Vol. 46, p. 355.

[2] Schwarzschild—Gott Nachrichten, p. 41, 1906.

complete at 10^{-3} atmospheres, and the heights shown by the lines of the unionised atoms of these elements are still lower. Compare table (2).

The results of the flash-spectrum observations are thus seen to be very satisfactorily accounted for on the basis of our theory.

Laboratory experiments also, as far as they go, are in qualitative agreement with our theory. It is well-known that in the flame, the lines due to the ionised atom, either do not occur at all, or even if they occur, they are extremely faint compared to the lines of the unionised atom. As the temperature is increased, the "enhanced lines" begin to strengthen, until at the temperature of the arc, they are comparable in intensity to the lines of the normal atom.

We give below the results of King[1] on the relative intensity of the 'enhanced' and ordinary lines of the alkaline earths in the vacuum tube furnace at varying temperature. Unfortunately the pressure which is a vital point, is not mentioned. The last line shows the percentage of the ionised atoms under a pressure of 10^{-4} Atmospheres, or ·7 mms. of mercury.

TABLES.

Element ...	Temp. Actual	1923°	2273°	2623°	Arc	Sun	
	Approximate	2000	2500	3000	4000	7500—6000 Rowland—Mitchell	
	Line.	Intensity.					
Ca ...	4227(g)	·300	500	1000	500	20	25
Ca^+ ...	3968(H)	18	25	50	350	700	80
	3933(K)	20	30	60	400	1000	100
Proportion of ionised atoms in per cents.		10^{-3}	$1{\cdot}6{\cdot}10^{-2}$	$7{\cdot}10^{-1}$	19	70 ($P=10^{-1}$)	91 $P=10^{-3}$
Sr ...	4607	300	400	600	600	1	2
Sr^+ ...	4215	6	15	30	400	5	40
	4077	12	25	40	400	8	40
Proportion of Sr^+ in per cents.		$35{\cdot}10^{-3}$	$1{\cdot}4{\cdot}10^{-1}$	5	36	80 ($P=10^{-1}$)	96 ($P=10^{-3}$)
Ba ...	5536	400	500	1000	1000	2	1
	4934	50	60	70	700	7	12
Ba^+ ...	4554	70	80	100	1000	8	20
Proportion of Ba^+ in per cents.		$2{\cdot}10^{-2}$	$7{\cdot}10^{-1}$	5	68	91 ($P=10^{-1}$)	98 ($P=10^{-3}$)

[1] Ladenburg and Lorin—Ber. d. D. Phys. Gesellschaft, 1098.

The tables show that an increase of temperature causes an increase of ionisation and the number of emission centres of the enhanced lines. The increasing intensities of the double lines are mainly to be ascribed to this fact. These become comparable in intensity to the principal lines of the normal atom, only when the degree of ionisation is rather large (comp. the figures at 4,000°K). Comparing the relative intensities of the corresponding lines of the Calcium, and Barium group, we find that for the same temperature the enhanced lines of Barium are relatively stronger than Calcium lines; and this, according to our theory, is due to the comparatively lower ionisation potential of Barium.

The objection may be raised whether the proportion of ionised atoms at low temperatures, as given by the theory, is not rather too low. The tables show that at 2,000°K, only 1 in 10^5 Calcium atoms is ionised. Is this small number of ionised atoms capable of affecting the photographic plate by the emission of the H, and the K lines?

No definite answer can be given to this point. We may however, point out that according to Ladenburz and Loria, when a Hydrogen vacuum tube at a pressure mms. of mercury is excited by a spark only 1 atom in 50,000 are found to be radiant, *i.e.*, capable of emitting the H_α and H_β. A very low proportion of radiant centres may therefore affect the photographic plate. It should also be remembered that at low temperatures the principal lines of the normal atom is not only relatively more intense, but very broad and diffuse, when the enhanced lines are extremely narrow, in addition to being faint.

A reduction in pressure will cause the relative intensity of the H-K lines to increase, but not the absolute intensity, because the total available number of radiant particles will now decrease. I am not aware whether there is any laboratory experiment for testing this point.

Hydrogen in the Sun.

It has been mentioned in the introduction that Hydrogen is not appreciably ionised at even the highest levels of the solar chromosphere. We should add to this the fact that hydrogen exists in the sun only in the atomic state, for if there were molecular hydrogen in the sun, we could have detected some at least, of the lines of the secondary spectrum. But this is not the case. Hydrogen enters into

chemical combination with Calcium, and Magnesium in the spot, but does not probably form molecules of its own.

We shall consider in this section whether these facts are reconcilable with our theory. This requires a knowledge of the heat of molecular combination and the ionisation potential of hydrogen.

These data already lie available in a recent paper by Frank[1] and others. They find evidences of the following chemical electron reactions.

$$H_2 = H + H + 3{\cdot}5 \text{ Volts } (=84{,}000 \text{ Cal.}) \quad \ldots \quad (A)$$

$$H_2 = (H_2)_+ + e + \quad \ldots \quad \ldots \quad \ldots \quad (B)$$

$$H = H_+ + e + 13{\cdot}6 \text{ Volts} \quad \ldots \quad \ldots \quad \ldots \quad (C)$$

$$H_2 = H + H + 2e + 30{\cdot}7 \text{ Volts } (=7{\cdot}1 \times 10^5 \text{ Cal.}) \quad \ldots \quad (D)$$

The first is a purely chemical reaction and the heat of molecular combination has been directly measured by Langmuir[2] and found to be 82,000 Calories. The ionisation voltage 13·6 in (C), can be calculated from the relation $h\nu_0 = \bar{e}V$ taking ν_0 = convergence frequency of the Lyman series, $\nu = N\left[\frac{1}{1^2} - \frac{1}{m^2}\right]$, *i.e.*, N. The actual occurrence of this process in the ionisation of H is indicated by a sudden increase of ionisation at 17·1 volts (13·6 + 3·5 volts). The ionisation voltage 30·7 in process (D) = 2. 13.7 + 3·5, corresponding to the complete breaking up of the H molecule into 2 atoms, and of these again into the core and the electron.

Let us first consider reaction (A),—dissociation of the molecule into atoms. Taking the equation of the reaction isobar

$$\log K = -\frac{U}{4{\cdot}571\,T} + \frac{\Sigma\nu C_P}{R}\log T + \Sigma\nu C$$

We have $$\Sigma\nu C_P = 2\left(C_P\right)_H - \left(C_P\right)_H - \left(C_P\right)_{H_2} = \frac{3R}{2}$$

$$C_H = -1{\cdot}6 \qquad C_{H_2} = -3{\cdot}40$$

We have $$\Sigma\nu C = {\cdot}20$$

$$U = 8{\cdot}2 \times 10^4 \text{ calories}$$

[1] Frank and others—Ber. d. D. Phys. Gesellschaft, Vol. 20.

[2] Langmuir—Zeits f Elektro chemie, Vol. 17.

$K = \frac{x^2}{1-x^2} P$ where P = total pressure, and a fraction x has been dissociated. We have thus

$$\log \frac{x^2}{1-x^2} P = -\frac{8{\cdot}2 \times 10^4}{4{\cdot}571\ T} + 1{\cdot}5 \log T + {\cdot}20.$$

Table 7 shows the dissociation of hydrogen under different pressure and temperature.

Dissociation of the H_2 molecule

Pressure	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Temper.					
2000	:	3	9	29	70
2500	11	35	75	97	
3000	46	85	98·5		
3500	86	98·5			
4000	96·5				
5000	100				

The table shows that in the sun, the dissociation is complete. Even in the umbra of sunspots, if we assume that the temperature is 4000°K and the pressure of the order of 1 atmosphere, the dissociation should be complete.

Ionisation of Hydrogen.

For a rigorous treatment of the case, we should start with the process (D). But since in the sun, the hydrogen is entirely in the atomic state, we may use the process (C). The results will but be approximate, for the equation which follows does not hold over the whole range of temperature.

The case is quite analogous to the ionisation of Calcium. We have only to put $U = 3{\cdot}2 \times 10^5$ Calories corresponding to 13·6 volts.

We have then

$$\log \frac{x^2}{1-x^2} P = -\frac{3{\cdot}2 \times 10^5}{4{\cdot}571 T} + \frac{5}{2} \log T - 6{\cdot}5$$

$$= \bar{6}{\cdot}\,020 \text{ for } T = 7500$$

$$= \bar{9}{\cdot}\,279 \text{ for } T = 6000.$$

These figures show that at a point where T=6000°, hydrogen can be completely ionised if $P=10^{-11}$ Atmospheres. Thus only at the highest points of the chromosphere, where the partial pressure falls to 10^{-11} Atmospheres, the ionisation can be complete, and we can expect the vanishing of the H-lines.

Helium.

The previous work on the ionisation of H-atoms will have made it clear that the higher the ionisation potential of an element the less will be its degree of ionisation under a given thermal stimulus. This is best exemplified in the case of Helium, which has got the highest ionisation potential of the elements so far investigated.

The experimental results, however, are rather discordant. According to Bohr. the ionisation potential should be 29 volts, while most investigators have detected the commencement of a distinct ionization at 20·5 volts. Some investigators have detected two distinct stages of ionisation, one at 20·5 and another at 25 volts. In addition to this last Rall detected a rather strong ionization when the potential is raised by 54·6 volts, *i.e.*, to about 80 volts.

The processes probably take place according to the following schemes:—

$$He = \overset{+}{He} + V_1 (=25{\cdot}6) \quad \ldots (A)$$

$$\text{Par } He = \overset{+}{He} + V_2 (=20{\cdot}5) \quad \ldots (B)$$

$$He = \text{par } He + V_1 - V_2 (=5{\cdot}1) \quad \ldots (C)$$

$$He + = \overset{++}{He} + V_3 \ (53{\cdot}4) \quad \ldots (D)$$

The distinction between He and the so-called parhelium is taken as one of relative configuration of the steady orbits of the two electrons. V_2 may be identified with 20·5 Volts, V_1 with 25 Volts, and V_3 with $4 \cdot 13{\cdot}6 = 54{\cdot}6$ Volts.[1]

Taking $V_2 = 20{\cdot}5$ Volts, $U = 4{\cdot}8 \cdot 10^5$ Calories approximately, we have the following table for the first step ionization of helium, If

[1] It is not possible to deduce V from the quantum relation eVh $(1, s)$, for the fundamental term $(1. s)$, is unknown both for helium and parhelium. What are generally called the Principal series of helium are really the series $\nu = (2, s) - (m, p)$ (the leading lines being 10380 for He, and 20587 for parhelium).

the ionization voltage be taken = 25 Volts, the degree of dissociation will become still less.

Ionisation of Helium (in per cents).

Press.	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Temp.						
6000	$5\cdot10^{-6}$	$1\cdot7\cdot10^{-6}$	$5\cdot10^{-5}$	$1\cdot7\cdot10^{-4}$	$5\cdot10^{-4}$	$5\cdot10^{-3}$
7000	$1\cdot10^{-4}$	$3\cdot6\cdot10^{-4}$	$1\cdot10^{-3}$	$3\cdot6\cdot10^{-3}$	$1\cdot10^{-2}$	$1\cdot10^{-1}$
7500	$4\cdot10^{-4}$	$1\cdot2\cdot10^{-3}$	$4\cdot10^{-3}$	$1\cdot2\cdot10^{-2}$	$1\cdot2\cdot10^{-1}$	...
8000	$1\cdot2\cdot10^{-3}$	$3\cdot7\cdot10^{-3}$	$1\cdot2\cdot10^{-2}$	$3\cdot7\cdot10^{-2}$	$1\cdot2\cdot10^{-1}$	$3\cdot7\cdot10^{-1}$
10000	$3\cdot10^{-2}$	$1\cdot10^{-1}$	$3\cdot10^{-1}$	1	3	31
11000	$1\cdot10^{-1}$	$3\cdot4\cdot10^{-1}$	1	34	11	72
12000	$3\cdot10^{-1}$	1	3	10	28	93
13000	$7\cdot10^{-1}$	2	7	22	58	...
14000	1·5	4	15	43	83	
15000	3	10	28	68	94	
16000	6	17	47	85	Complete Ionisation	

Thus the above table does not seem to favour the conclusion that helium should get ionized in any appreciable quantity any where in the solar atmosphere. It also shows that He^+ can occur in appreciable quantity only in stars having a temperature of not less than 15000° K. These problems will be attempted in a separate communication.

Besides the lines of Ca^+, Sr^+, Ba^+ and Mg^+, the chromosphere is distinguished by lines of Fe^+, Sc^+, Ti^+, and Va^+. But as the spectroscopic properties of these elements have not yet been investigated, no attempt has been made in the present paper for the investigation of these elements from the present standpoint. But it seems to be fairly certain that the ionisation potential of these elements lie between 5 to 7 Volts.

On the Application of Electron-Chemistry to Problems of Radiation and Astrophysics.[1]

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Both classical and modern Thermodynamics have hitherto been confined to the treatment of the influence of heat on material substances up to the stage of chemical decomposition and vaporisation (or atomisation). The successive stages which comes up for treatment can be thus schematically written.

Phase.	Phenomenon.	Example.	Temperature.
Solid ...	... ...	Ice	$<273°K$
↓	Liquefaction		
Liquid ...	... ...	Water	$<373°K$
↓	Vaporisation		(1 Atm)
Gas (consisting of molecules) ...	... ...	Steam (H_2O)	$>373°K$
↓	Decomposition		(1 Atm)
Gas (constituent molecules) ...	... ...	H_2, O_2	
↓	Atomisation		
Gas (Elementary Atoms).		H. O	$>3500°K$

What happens when the gaseous-mass, consisting purely of atoms, is further heated? The problem has not merely an academic interest, for though the temperatures we are considering may not be commanded in the Laboratory, such is usually the case in the stellar universes with which we are acquainted through their spectra.

[1] An introduction and synopsis of the following four-papers: —

Paper A Ionisation in the Solar Chromosphere.
B On the Problems of Temperature-Radiation of Gases.
C On Elements in the Sun.
D On the Harvard Classification of Stellar Spectra.

The answer to the problem raised easily follows from considerations of the Rutherford-Bohr theory of the atom, *viz.*,—futher heating of the gaseous mass will cause ionisation, *i.e.*, some of the atoms will lose one electron, and under particular conditions of temperature and pressure, a definite chemical equilibrium will be established between the neutral atoms, the atoms which have lost one electron, and the electrons split off, according to the scheme

$$Ca \underset{\longleftarrow}{\longrightarrow} Ca^{+} + e \qquad \ldots \quad (1)$$

That this will be the case may be seen from the fact that at high temperatures, many metals throw off copious quantities of electrons. In these substances (tungsten for example), ionisation precedes liquefaction, just as in camphor, carbon and other volatile substances, vaporization (sublimation) precedes liquefaction.

These problems were foreshadowed by Nernst in this book 'Das Neue Wärmesatz'; in page 154, we come across the following passage :—

"Die thermische Dissoziation eines Atoms in das positive Ion und das negative Elektron ist also eindeutig bestimmt, wenn wir die Dissoziationwärme kennen. Letzere ist in gewissen Fällen durch das Bohrsche Atommodell gegeben, wenn freilich dieser Weg zuzeit auch noch einigermassen hypothetisch ist."

The equation of chemical equilibrium of the process (1) is given by Nernst's law of Reaction-Isochore,

$$\log k = \log \frac{x^2}{1-x^2} \quad P = -\frac{U}{2\cdot 3RT} + \log T \frac{\Sigma\nu C_p}{R} + \frac{\Sigma\nu C}{R}.$$

Where x is the fraction of the total number of atoms dissociated.

U = heat of ionisation, in the energy-relation

$$Ca = Ca^{+} + e - U$$

$\Sigma\nu C_p$ = Sum of the atomic specific heats of the reacting atoms including the electron, taken with the proper algebraic sign. $\Sigma\nu C$ = Sum of the chemical constant of the reacting atoms including the electron, taken with the proper algebraic sign.

As suggested by Nernst, the specific heat, at constant pressure, of the electron is $\frac{5}{2}$ R (assuming the electron to be a monatomic gas) and its chemical constant can be calculated from the Tetrode-Sackur formula :—

$$C=-1\cdot6+\tfrac{3}{2}\log M\,;\; i=C+\log(1\text{ Atm})=\frac{(2\pi m)^{\frac{3}{2}}k^{\frac{5}{2}}}{h^3}.$$

If we put M, the atomic weight of the electron $=\frac{1}{1836}$, Eggert assumes the chemical constant of C_a and C_i to be equal.

The formula finally stands as

$$\log\frac{i^2}{1-i^2}\;P=-\frac{U}{2\cdot3RT}+\tfrac{3}{2}\log T-6\cdot5.$$

The value of U still remains undetermined. At this place, Nernst's suggestion is entirely misleading, for the ionisation-potential of elements as determined by Franck and Hertz, MacLenan and others gives us the most exact data for the calculation of U, and it is unnecessary to introduce any artificial hypothesis for calculating U, as Eggert [1] has done.

The ionization-potential gives us the amount of energy which is required for tearing the outer-most electron from an atomic system to infinity. The quantity U (heat of ionisation for a gm-atom) can therefore be calculated from the relation,

$$U=\frac{eV.N}{300J}\qquad \begin{aligned} &N=\text{Avogadro Number.}\\ &J=\text{Mechanical Equivalent of Heat.}\end{aligned}$$

In cases where the ionisation-potential is not known, its value can be calculated from spectroscopic data from the value of $(1, s)$ which is the convergence-frequency of the principal series of the element, according to the quantum-relation,

$$V=\frac{h(1,s)c}{\quad}\times300.$$

[1] Eggert's object was to verify Eddington's hypothesis of ionisation in the interior of stars.

With the aid of equation (1), we can calculate and tabulate the degree of ionisation of the following elements at definite temperatures and pressures :—

Mg, *Ca*, *Sr*, *Ba*—*Na*, K, *Rb*, *Cs*,—H, *He*,. For other elements we have no satisfactory data.

Tables of percentage ionisation have been prepared from formula (2′), and successfully applied to the following astrophysical problems.

(1) *Occurrence of Elements in the sun and stars.*

It has been shown that the alkali metals *Rb*, *Cs* are almost completely ionised in the sun, and hence cannot be detected by their ordinary lines. The lines of their ionised atoms lie in the ultraviolet. *k* (I. P=4·32 Volts) is 80% ionised, and hence is only feebly represented. *Na* (I. P=5·12 Volts) is 60% ionised on the photosphere, but the ionisation becomes complete when the pressure falls to 10^{-3} to 10^{-4} Atoms, and hence the D-lines are confined to the lower layers of the chromosphere. In the sunspots, temperature being diminished to about 5000°*k*, the ionisation falls down to 5% and hence the D-lines are greatly intensified.

(2) *The Occurrence of Enhanced lines in the chromosphere.*

of the alkaline earths, *Ca* is 30% ionized, while *Sr* and *Ba* are 50% and 60% ionised respectively. Hence on the photospheric level, we get not only the lines of *Ca*, *Sr*, and *Ba* but also of Ca^+, Sr^+, and Ba^+, but at greater heights, owing to the fall of pressure, ionisation becomes almost complete. This explains the disappearance of the *g*-line of *Ca* and of the corresponding lines of *Sr* and *Ba*, and the exclusive occurrence of the Enhanced lines in the high-level chromosphere. The case of the *Mg*-lines is also successfully treated.

With the aid of the following formula which represents the degree of dissociation of a diatomic molecule

$$\log \frac{x^2}{1-x^2} P = -\frac{U}{2{\cdot}3RT} + \frac{3}{2}\log T + \Sigma rC$$

it is shown, that H_2 and O_2 are completely dissociated into atoms not only in the sun, but also in sunspots, and are represented by their atomic spectra only. N_2 may remain partly undissociated and hence show its molecular spectrum (the so-called cyanogen bands).

A method for dealing with the spectra of chemical compounds is also foreshadowed.

(3) It is shown from considerations of the Bohr-Sommerfeld theory of spectral emission that the transition of an atom from a neutral state to the ionised state is not abrupt, but is marked by successive states of equilibrium of the atomic system. In the normal state, the system has the energy $A-h(1, s)$, corresponding to the possession of one quanta of angular momentum by the outermost vibrating electron. The next stable states of the atom are marked by the energy-contents $A-h\ (2, p)$, $A-h\ (3, d)$... etc...corresponding to the possession of 2, 3,...m radial quanta of angular momentum by the vibrating electron. Ionisation corresponds to $m=\infty$

It therefore follows that radiation of normal lines will precede ionisation, and the order in which the lines will come out, is as follows :—

$$(1, s)-(m, p),\ (2, p)-(m, d),\ (3, d)-(m,b),\ etc.$$

$$(2, s)-(m, p),\ etc.$$

The temperature of emission of a certain group of lines therefore bears a relation to the temperature of ionisation of the atom, the general rule being that the higher the temperature of complete ionisation, the higher is the temperature at which the gas can be made to glow. It was shown that the existing data on radiation of elements under a purely thermal stimulus can best be explained on the above hypothesis. Thus while elements like Ca or K possessing a low ionisation potential can be excited even in the flame, even the highest temperatures fail to excite H_2, O_2, N_2, A which have very high ionisation-potentials varying from 13·6 to 25 volts. In the case of hydrogen, the temperature of complete ionisation is about 22000°k, and the temperature of emission for the Balmer lines is not less than 4500°k. The corresponding figures for Helium are 35000°k and 12000°k.

(4) *The second stage ionisation of elements.*

If P=1 Atms, Ca becomes completely ionised when T=13000°.k

Beyond this stage, we have only Ca^+ atoms. But this now begins to

get further ionised. A theory of this second-step ionisation is also worked out. The formula is approximately

$$\log \frac{x^2}{(2+x)(1-x)} P = -\frac{U}{2\cdot 3RT} + \tfrac{5}{2}\log T - 6\cdot 5.$$

With the aid of this formula, the second-step ionisation of *Ca*, *Sr*, *Ba*, *Mg*, *He* have been calculated.

The following table contains the results of the application of the above formulae to problems of stellar spectra. A physical basis is thus provided for the Harvard classification of stars according to the nature of their spectra. Column 5 shows the stellar-class at which the phenomena described under column 1 takes place. This has been compiled from the Harvard Annals. The temperature under column 2 has been calculated from the formulae given in the papers.

Phenomena.	Temp.	Press.	Remarks.	Stellar-class.
Appearance of K	5000°	1 Atm	Beginning of the Ionisation of *Ca*	M*a*
Disappearance of *g*	13000°	,,	Ca completely ionised	B*g*
Appearance of *Mg* (4481)	7500	,,	Beginning of the Ionisation of *Mg*	G*o*
Disappearance of K	19000	,,	Ca completely ionised	O*d*
,, of 4481	23000	,,	*Mg*	O*a*
Appearance of *He*	16000	,, -5	Beginning of the Ionisation of *He*	B3 A
Disappearance of *He*	30000°	10	He completely ionised	P*e* (nebula)
Luminescence of H begins at	4500°	1 Atm.	Appearance of (2, p) orbit	M*o*
Luminescence of He begins at	12000°			A*o*
Maximum Luminescence of H	12000		Maximum concentration of atoms with (2, p) orbits	A*o*
Maximum Luminescence of *He*.	17000			B2A

The conclusion is therefore made "The continuous variation of stellar spectral types, as observed by the Harvard Astrophysicists, can mainly be ascribed to the varying value of the temperature of emission of the stellar atmospheres."

We can now furnish a complete scheme of the train of physical phenomena which we come across when the temperature is gradually enhanced.

Phase.	Phenomena.	Example.	Temp.
Solid	...	Ice	$<273°k$
	↓ Liquefaction		
Liquid	...	Water	$<373°k$
	↓ Vaporization		
Gas		Steam	$>373°k$
	Decomposition attended with the emission of H_2O spectrum. ↓		
Gas (molecules of the constituents).		H_2, O_2	
	Atomisation (Emission of the molecular Spectrum). ↓		
Gas (Atoms with $(1, s)$ orbits).		H	
	Emission of $(1, s)-(m, p)$ lines. ↓		
Gas (Atoms with $(2, p)$ orbits).		H	$5000°k$
	Emission of $(2, p)-(3, d)$ lines. ↓		
Gas (Atoms with $(3, d)$ orbits).		H	
	Ionisation (First Step) ↓		
Gas (Positively charged atom and the electron).		H^+	$22000°k$
		Ca^+	$13000°k$
	Ionisation, with radiation (Second Step), of enhanced lines.		
Gas (Atom with 2 plus charges and electron,		Ca^{++}	$19000°k$

etc., till in the interior of stars, the whole mass consists of positive nucleii, and electrons.

The theory probably also provides us with the long-sought-for source of stellar energy. It is well-known that neither the energy of

gravitational contraction nor the energy of radioactive disintegration suffices for this purpose. All observational data point out that the source is of the nature of latent heat,[1] and it is only one step to conclude this heat is provided when, owing to a fall of temperature, a fraction of ionised atoms combines with electrons producing the neutral atom, or the ionised atom with one plus charge less, with the liberation of latent heat of ionisation. The general trend of this source of heat will be to retard the progress of cooling, and it can come to action only at about $4000°k$.

[1] Compare Science Progress, April, 1920.



Fig.(1)

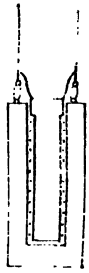


Fig (2)

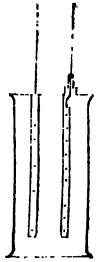


Fig.(3)

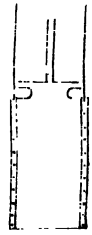


Fig. (4)

A Report on the Theories of the Specific Heat of Solids.

BY

Sasibhushan Mal, M.Sc.

Dulong and Petit's Law of Specific Heat for solids.

Since the time of Black many ingenious methods have been employed for the determination of the specific heat of substances. In the year 1819, two French physicists, P. L. Dulong and A. T. Petit, made the first attempts for establishing an empirical relation between the specific heats of various substances and as a result of their researches they established the well-known law that the product of the atomic weight and the specific heat is a constant for all elementary substances. Dulong and Petit gave the value of the constant as 6·4 for all substances which are in the solid state at ordinary temperatures.

The following table indicates how far Dulong and Petit's law is applicable in practice. The specific heats are all measured at constant pressure and at ordinary temperatures, and most of them are what we now call 'the mean specific heat.'

Table I*

Atomic number.	Atomic weight.	Element.	Atomic heat.	Atomic heat (Nernst.)
3	7	Lithium ...	6·6	
11	23	Sodium ...	6·7	
12	24·3	Magnesium ...	6	
13	27·1	Aluminium ...	6	5·98 (at 550°K)
15	31	Phosphorus (colourless)	6·2	
16	32	Sulphur (rhombic) ...	5·5	
19	39·1	Potassium ...	6·5	
20	40	Calcium ...	6	
22	48·1	Titanium ...	5·4	
24	52	Chromium ...	5·4	

* Most of the results of Tables I, II and III are taken from Sackur's Thermochemistry and Thermo-dynamics and from Kaye and Laby's Physical and Chemical constants.

Atomic number.	Atomic weight.	Element.	Atomic heat.	Atomic heat (Nernst).
25	54·9	Manganese ...	6·7	
26	55·8	Iron ...	6·6	
27	59	Cobalt ...	6·1	
28	58·7	Nickel ...	6·4	
29	63·6	Copper ...	5·9	5·87 (at 450°κ)
30	65·4	Zinc ...	6	
33	75	Arsenic (crystalline)...	6·2	
34	79·2	Selenium (crystalline)	6·6	
35	79·9	Bromine (solid) ...	6·7	
42	96	Molybdenum ...	7	
44	101·7	Ruthenium ...	6·2	
45	102·9	Rhodium ...	6	
46	106·7	Palladium ...	6·3	
47	107·9	Silver ...	5·9	5·99 (at 589°k).
48	112·4	Cadmium ...	6·2	
50	118·7	Tin ...	6	
51	120·2	Antimony ...	6·1	
52	127·5	Tellurium ...	6·1	
53	126·9	Iodine ...	6·9	
55	132·8	Caesium ...	6·4	
58	140·3	Cerium ...	6·3	
74	184	Tungsten ...	6·3	
76	190·9	Osmium ...	5·9	
77	193·1	Iridium ..	6 2	
78	195·2	Platinum ...	6·2	
79	197·2	Gold ...	6·1	
81	204	Thallium ...	6·7	
82	207·2	Lead ...	6·2	6·07 (at 409°k).
83	208	Bismuth ...	6·2	
92	238·2	Uranium ...	6·7	

The table clearly indicates that the elementary substances which are in the solid state at ordinary temperatures approximately obey Dulong and Petit's law. In fact, we cannot expect that the law should be obeyed quite accurately. For, we know from experiments that the specific heat of all substances (solid, liquid and gaseous) varies with temperature and also we know that it depends to some extent on the Physical state of the substance,—*e.g.*, the three varieties of carbon —graphite, diamond, and amorphous carbon—have different specific heats. It was Dulong and Petit themselves who first showed that the specific heat of all substances increases with temperature and this conclusion has been corroborated by all subsequent investigations.

In the case of most elements which are ordinarily in the solid state the variation of specific heat with temperature is very small except at very low temperatures, and these are the elements which approximately obey Dulong and Petit's law. But there are some solid elements of low atomic weight for which the variation of specific heat with temperature is very great even at ordinary temperatures, and these are exactly the elements which show the greatest deviations from Dulong and Petit's law. Boron, Carbon, Beryllium and Silicon are among these refractory elements. These elements are characterised by high melting points and with the increase of temperature their atomic heat (the product of the atomic weight and the specific heat) gradually approaches the limit 6·4. Hence, if instead of comparing the atomic heats of elements at room temperatures we compare them at what are known as "comparable temperatures,"* Dulong and Petit's law will be much more generally obeyed. The following table indicates the increase of atomic heat with temperature for the refractory elements :—

TABLE II

Substance.	Atomic weight.	Temperature.	Atomic heat.
Beryllium ...	9·1	0°c	3·4
		100°c	4·3
		300°c	5·3
		500°c	5·6
Boron	11·0	26°·6c	2·6
		233°c	4·03
		Red heat	5·5
Diamond ...	12·0	—53°c	0·72
		11°c	1·35
		206°c	3·3
		896°c	5·48
Silicon (Crystalline)	28·3	21°·6c	4·8
		232°·4c	5·75

* *i.e.*, Temperatures having the same ratio to their melting points, for example, the melting points of lead and copper are 600°k. 1,313°k respectively, so they are at comparable temperatures at 27°c, 383°c respectively.

In the case of the refractory elements not only that the value of Dulong and Petit's constant is different for different elements at ordinary temperatures, but even for the same element which exists in more than one allotropic modification the atomic heat is different for the different modifications at the same temperature. Thus in the case of carbon we find that the atomic heats of wood charcoal, gas coal and graphite—all of which are allotropic modifications of carbon—are quite different from one another. The following table indicates this for a few elements :—

TABLE III

Substance.	Allotropic modification.	Temperature in degree centigrade.	Atomic heat.
Carbon	Diamond	11	1·35
	Native graphite	22—70	2·2
	Furnace graphite	22—70	2·2
	Gas carbon	24—68	2·47
	Charcoal	0—24	2
Boron	Crystalline	0—100	2·77
	Amorphous	0—100	3·38
Silicon	Crystalline	21	4·7
	Amorphous	21	6·06
Sulphur	Rhombic	0—54	5·54
	Monoclinic	0—52	5·80
	Amorphous (insoluble)	0—53	6·10
	Amorphous (soluble)	0—50	8·0

So long we considered only the specific heat of solids at constant pressure. But if the specific heat at constant volume be considered, it will be found that Dulong and Petit's law is much better obeyed. In fact, Nernst and Lindemann tested the law by calculating the atomic heat not at constant pressure but at constant volume (more later on).

Extension of Dulong and Petit's Law to compounds.

Woestyn suggested an extension of Dulong and Petit's law of specific heat for elements to compounds on the supposition that the atomic specific heat of elements remains conserved even when the latter enter into chemical combination. In practice it is found that this applies approximately to the case of metallic alloys whose temperatures are much below their fusion points. In the year 1831 F. E. Neumann propounded a law of specific heat for compounds, namely that for

compounds which are similarly constituted and which have the same general formula, the product of the molecular weight and the specific heat is a constant, but the value of the constant varies from one series of compounds to another. Neumann's law, which was subsequently confirmed by Regnault, is but a rough representation of facts. To obtain a law which represents facts better Kopp made the assumption that the atomic heats of all elements are not equal to one another, and he gave the law that the molecular heat of a compound is the sum of the specific heats of the individual atoms forming the molecule. Kopp calculated the specific heat of many elements from experimental data on the specific heat of compounds. For example, he obtained 2·7, 1·8, 2·3, as the atomic heats of boron, carbon and hydrogen respectively. But Kopp's law does not take into account the variation of specific heat of atoms and molecules with temperature, nor does he consider the forces between atoms in a molecule, and hence Kopp's law for the specific heat of compounds, though it represents facts better than Neumann's law, is bound to fail ultimately. In fact, Kopp's law for compounds does not hold even as accurately as Dulong and Petit's law for elements.

Theoretical Explanation of Dulong and Petit's Law—L. Boltzmann.

The theoretical explanation of Dulong and Petit's law was first given by Ludwig Boltzmann from the standpoint of the kinetic theory of matter. He supposed that the heat energy of a solid substance is made up of the energies of the individual atoms. In the solid state the atoms can only have motions of vibration about a mean position of equilibrium and all its vibrations can be resolved along three cardinal directions, that is, it has three degrees of freedom. The total energy of vibration of an atom for each degree of freedom is half kinetic and half potential. According to the well-known theory of equipartition of energy of Clerk Maxwell and Boltzmann, the mean kinetic energy of a moving particle at temperature θ is $\frac{1}{2} k\theta$ for each degree of freedom, where 'k' is Boltzmann's universal gas constant. Hence, the mean kinetic energy of a vibrating atom at temperature θ is $\frac{3}{2} k\theta$ and the corresponding mean potential energy is $\frac{3}{2} k\theta$, so that the total energy of the vibrating atom at temperature θ is $3k\theta$. If N be the Avagadro number (the number of atoms in a gram atom) the total energy of vibration E of the gram atom is $3Nk\theta$ or $3R\theta$ where R is the universal gas constant.

Thus the atomic heat $= mC_v$.

$$= \frac{d}{d\theta} E.$$

$$= \frac{d}{d\theta} (3 R\theta)$$

$$= 3R$$

$$= 5{\cdot}96 \text{ calories}$$

Since $R = 8{\cdot}30 \times 10^7$ erg/degree

Hence, atomic heat $= 1{\cdot}984$ cal/degree.

From table I it is apparent that the atomic heat of almost all metals slightly exceeds the theoretical value (5·96 calories) given by Boltzmann. It is partly due to the fact that the specific heats given in Table I are specific heats at constant pressure. Nernst calculated the specific heat of some metals at constant volume by making use of a method to be indicated hereafter and his values are in better agreement with Boltzmann's theoretical value.

It was shown by the experiments of Ben Tilden and Dewar that the specific heats of most metals decrease rapidly at low temperatures. This explains the behaviour of the refractory elements,—carbon, boron, silicon and beryllium,—which at ordinary temperatures are in a state corresponding to the state of these metals at sufficiently low temperatures. In 1906 Nernst showed from his 'Heat Theorem' that the specific heat of all substances should become null near about the absolute zero. But according to Boltzmann's theory which is directly based on the Maxwell-Boltzmann theory of equipartition of energy, the specific heat of solids should be the same at all temperatures. Thus, we find that the equipartition theory fails to explain the specific heat of solids. This anomaly was afterwards satisfactorily explained by invoking the aid of the well-known quantum hypothesis of Planck.

The pioneer in this direction was A. Einstein, whose first papers appeared in the year 1907.* The theory of equipartition of energy tacitly assumes that the energy of vibration of a particle may change by any infinitesimally small amount. According to the quantum hypothesis which is a denial of the theory of equipartition of energy, the energy of vibration of a particle at any moment cannot have any

Works of Einstein on the specific heat of solids, 1907.

* Ann. d. Physik, 22, 180 (1907).

random value but its value must always be an integral multiple of a quantum of energy $h\nu$* which is characteristic of the vibrating molecule and is proportional to its characteristic frequency of vibration ν, h being a constant known as Planck's universal constant. Thus, the energy of vibration of a particle can change only by quanta characteristic of the particle. Now, suppose that in a chamber in temperature equilibrium at the temperature θ, the distribution of the vibrating particles (or resonators as Planck called them in analogy with the Hertzian resonators) each having a characteristic frequency ν is given by the law†

$$N_p = N_0 \, e^{-\frac{ph\nu}{k\theta}}$$

where N_0 is the total number of particles each having zero energy, N_p is the total number of particles each of which has p quanta of energy corresponding to each degree of freedom and p is any integer.

The total number of resonators in the volume considered is given by

$$\begin{aligned} N &= N_0 + N_1 + N_2 + N_3 + \cdots + N_p + \cdots \\ &= N_0 + N_0 \, e^{-\frac{h\nu}{k\theta}} + N_0 \, e^{-\frac{2h\nu}{k\theta}} \cdots + N_0 \, e^{-\frac{ph\nu}{k\theta}} + \cdots \\ &= N_0 \left(1 + e^{-\frac{h\nu}{k\theta}} + e^{-\frac{2h\nu}{k\theta}} + \cdots + e^{-\frac{ph\nu}{k\theta}} + \cdots\right) \\ &= \frac{N_0}{1 - e^{-\frac{h\nu}{k\theta}}} \end{aligned}$$

The total energy of all the resonators in the same volume is given by

$$\begin{aligned} E &= 0 \times N_0 + h\nu \, N_1 + 2h\nu \, N_2 + \cdots + ph\nu \, N_p + \cdots \\ &= h\nu \cdot N_0 \, e^{-\frac{h\nu}{k\theta}} + 2h\nu \, N_0 \, e^{-\frac{2h\nu}{k\theta}} \cdots + ph\nu \, N_0 \, e^{-\frac{ph\nu}{k\theta}} + \cdots \\ &= N_0 h\nu \frac{e^{-\frac{h\nu}{k\theta}}}{\left(1 - e^{-\frac{h\nu}{k\theta}}\right)^2} \end{aligned}$$

* For a historical account of the rise and growth of the quantum theory, see Jeans' Report on the Quantum Theory and Radiation.

† This method of calculating the average energy from the standpoint of the Quantum Theory is due to Poincare— Les Theorie du Rayonnement et les Quanta.

Hence the mean energy of the resonator for each degree of freedom is

$$\frac{h\nu e^{-\frac{h\nu}{k\theta}}}{1-e^{-\frac{h\nu}{k\theta}}}$$

$$= h\nu . \frac{1}{e^{\frac{h\nu}{k\theta}}-1}$$

$$= k\theta . \frac{x}{e^{x}-1}$$

where, $x = \frac{h\nu}{k\theta}$.

We thus prove that for each degree of freedom the mean energy of vibration of a particle at the temperature θ and having a characteristic frequency of vibration ν is

$$k\theta \frac{x}{e^{x}-1} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Planck gives another expression for this energy namely

$$k\theta \frac{x}{e^{x}-1} + \frac{h\nu}{2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

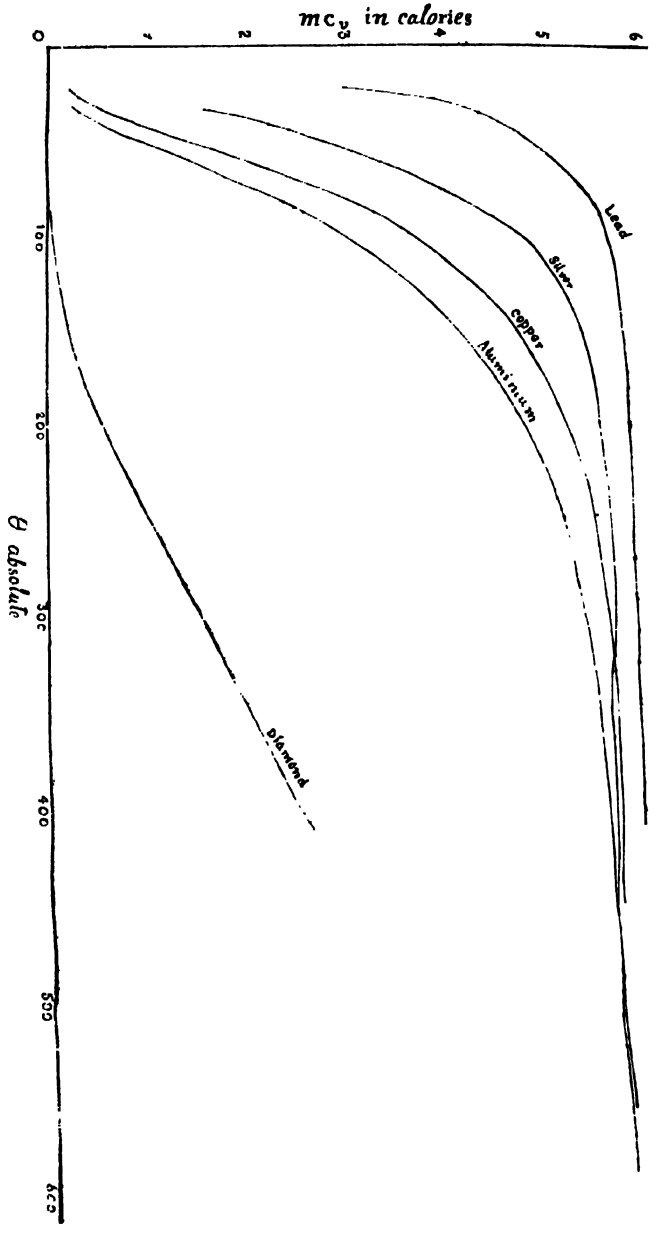
Formula (1) is known as Planck's first formula for the energy of a vibrating particle for each degree of freedom and formula (2) is his second formula. Now, after the manner of Boltzmann, Einstein assumed that in a solid each atom is vibrating about a mean position, and he made the further assumption that in an element all the vibrating atoms or resonators have one single frequency characteristic of the nature of the element. A vibrating atom in a solid has three degrees of freedom, and for each degree of freedom the mean energy possessed by the atom is that given by Planck's law. Hence, the total energy of the gram-atom of the solid is given by the formula

$$E = 3\,N\,K\theta \frac{x}{e^{x}-1}$$

$$= 3R\theta \frac{x}{e^{x}-1}$$

$$\text{or } E = 3\,R\theta \frac{x}{e^{x}-1} + \frac{N h\nu}{2}$$

* The second formula is deduced by a different method from the theory of Black-body Radiation; *vide* Planck, Wärme-Strahlung, p. 140.


mc_v in calories
0
1
2
3
4
5
6
θ absolute
100
200
300
400
500
600
Lead
Silver
Copper
Aluminium
Diamond

It is easy to see that when $\theta=0$, $E=0$ according to the first formula and $E=\frac{Nh\nu}{2}$ according to the second formula. So, according to the first formula there is absolutely no energy at the null point of temperature while according to the second formula there is a residual energy $\frac{Nh\nu}{2}$ at the null point which is known as Einstein's null-point energy. But for the Specific Heat, we get the same value from both the formulæ.

$$mc_v=-\frac{dE}{d\theta}=3R\frac{x^2e^x}{(e^x-1)^2}$$

where $x=\frac{h\nu}{k\theta}$

Suppose $\frac{h\nu}{k}=\theta_m$ so that $x=\frac{\theta_m}{\theta}$

Hence, $mc_v=3Rf\left(\frac{\theta_m}{\theta}\right)$.

At high temperatures x is small and hence $f(x)=1$. At low temperatures x is large, so that $f(x)=3Rx^2e^{-x}=$ a very small quantity. Hence, according to Einstein's formula the atomic specific heat of solids at high temperatures tends to the value 3R and at low temperatures it tends to the value zero.

The characteristic frequency ν of a solid substance determines the characteristic temperature θ_m for that substance, and if θ_m is once known the value of the function $f(x)$ for that substance becomes known and its specific heat at all temperatures can at once be calculated.

One of the deductions from Einstein's theory of specific heat for solids is that if we compare the specific heat curves of two solids there will be found a point-to-point correspondence in the two curves. If a line be drawn parallel to the temperature axis cutting the two curves at two points the abscissae of these two points will always have the same ratio no matter in whatsoever part the line may have been drawn. This point was afterwards tested by Nernst and was found to tally with facts.

In 1911 appeared the researches of Nernst and his co-workers on the specific heat of solids at low temperatures.* The modern methods of determining specific heats are almost entirely due to Nernst and his co-workers. As early as 1906 Nernst had invented the vacuum calorimeter which is a great improvement upon the previously existing forms of calorimeters. With the vacuum calorimeter the specific heat can be measured very accurately within very narrow regions of temperature and this enables us to trace the variations of specific heat with temperature in a very satisfactory manner.

Experimental work of Nernst and his co-workers.

The principle of the vacuum calorimeter is that the solid whose specific heat is to be determined acts as its own calorimeter. It is heated by passing a current through a platinum wire surrounding it and the change in temperature is measured by the change in resistance of the wire itself. The amount of energy added is obtained from the current supplied to the wire and the potential difference between the ends of the wire.†

Fig. (1) represents the arrangement of a vacuum calorimeter. The calorimeter C is suspended by wires which are connected to the platinum heating coil inside it. The calorimeter itself is placed in a glass vessel which is carefully evacuated with a Gaede Pump. The resistance of the platinum wire and the energy supplied to the wire are to be measured with precision ammeters and voltmeters.

There are principally three types of vacuum calorimeters. The first two types are used for work down to the temperature of liquid air and the third type is used for work below liquid air temperatures.

Fig. (2) represents the calorimeter for determining the specific heat of substances which are good conductors of heat (*e.g.*, metals) and it can be conveniently used up to the temperatures of liquid air. A hole is bored in the centre of a cylindrical block of the substance whose specific heat is to be determined and into this hole is fitted a block of the same substance having a platinum wire wound round it. The top of the inner block is somewhat thicker than the lower parts so that it may closely fit into the hole and ensure good contact. The platinum wire is insulated by using paraffined paper and the space between the two blocks is also filled with paraffin.

* Nernst, Ann. d. Phys. 36, 395 (1911).

† The vacuum Calorimeter in its details was worked out by Eucken in Nernst's laboratory in the year 1909.

Fig. (3) represents the calorimeter for determining the specific heat of substances which are not good conductors of heat, and this also may be used up to the temperatures of liquid air. The substance whose specific heat is to be determined is enclosed in a silver vessel. The platinum wire is wound on a silver tube placed inside the silver vessel and soldered to it, and shellac is used as the insulating material. One end of the platinum wire is soldered to the silver vessel and the other end passes through a small platinum tube soldered to the silver vessel and closed with fusible lead glass. The vessel is filled with some gas in order that temperature equilibrium may be quickly attained during an experiment.

Fig. (4) represents the calorimeter for work below the temperatures of liquid air. It is small in size and hence very convenient to use. It consists of a silver vessel round the outside of which the platinum wire is wound, and the wire is further protected by a mantle made of silver foil which is soldered to the silver vessel at the bottom-edge. This protects the calorimeter from loss of heat.

In using the calorimeter the glass bulb containing the calorimeter must first be filled with a gas such as hydrogen and placed in a bath at the temperature at which the specific heat is to be determined, and when the calorimeter has attained the temperature of the bath the gas is to be pumped out in order to ensure heat insulation

Nernst and his co-workers carried out a long series of experiments on the specific heat of solids at low temperatures. The formula they used for determining temperature from the change of resistance of platinum was an empirical formula obtained by determining the resistance of platinum at standard low temperatures. They determined what is known as the specific heat at constant pressure. The specific heat at constant volume can be determined from a knowledge of the specific heat at constant pressure with the help of the thermo-dynamical formula

$$C_p = C_v + \frac{9\lambda^2 TV}{k}$$

where C_p = Sp. ht. of a gram atom at constant pressure.
C_v = Sp. ht. of a gram atom at constant volume.
λ = Co-efficient of linear expansion.
T = Absolute temperature.
V = Volume of the gram atom of the solid.
and k = Compressibility.

But in the case of most substances the requisite data for the calculation of the specific heat at constant pressure from the value of the specific heat at constant volume are wanting. For this reason Nernst and Lindemann applied the approximate formula.

$$C_p = C_v + 0.0214\ C_p^2 \frac{T}{T_s} \ *$$

where T_s = melting points of the substance.

* This formula can be deduced as follows:—
According to Einstein,

$$\nu_m = 2 \cdot 54 \times 10^7 \frac{V^{\frac{1}{6}}}{m^{\frac{1}{2}} k^{\frac{1}{2}}}$$

where ν_m = Characteristic frequency of the solid.
m = Atomic weight of the solid.
V = Atomic volume of the solid.
and k = Compressibility of the solid.

But, according to Lindemann,

$$\nu_m = 2 \cdot 80 \times 10^{12} \sqrt{\frac{T_s}{m V^{\frac{2}{3}}}}$$

Where T_s = melting point of the solid. From Einstein and Lindemann's formula we have the relation.

$$\frac{V}{k} = 1 \cdot 21 \times 10^{10}\ T_s \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (a)$$

But, according to Mie, $\frac{3\lambda V}{kc_v}$ is a universal constant. Combining this with the equation (a) we find that 3λ is proportional to $\frac{C_v}{T_s}$

Hence,

$$C_p = c_v + \frac{9\lambda^2 TV}{k},$$

$$= c_v \left(1 + 3\lambda T \cdot \frac{3\lambda V}{kc_v}\right)$$

$$= c_v \left(1 + \frac{C_v T}{T_s} \cdot \frac{3\lambda V}{kc_v} p\right),$$

where p is a constant.

$$= c_v + c_v^2 \frac{T}{T_s} A_s,$$

where A_s is a constant.

Both $\frac{T}{T_s}$ and A_s are small quantities, so that $c_v^2 \frac{T}{T_s} A_s$ is a small quantity. Hence, to a first approximation, the formula may be changed to the form

$$C_p = C_v + C_p^2 \frac{T}{T_s} A_s.$$

The value of A_s is taken to be 0·0214.

The above expression has recently been simplified into the form

$$C_p = C_v + \alpha T^{\frac{3}{2}}$$

where 'α' is a constant characteristic of the substance considered. (See Nernst's Theory of Solid State.)

With the best available data then existing Nernst calculated the value of c_v for several substances by making use of the two formulae mentioned above. Some of his results are shown in column 3 of Table I.

The empirical results obtained by Nernst and his co-workers can be summarised as follows.

(1) At very low temperatures the atomic heat of elements tends to the value zero and the temperature—specific heat curve approximately touches the temperature axis. This means that at a finite temperature the specific heat of the substance is vanishingly small. For example, in the case of diamond the specific heat was found to be 0·000 at 70°K, and in the case of aluminium the specific heat was found to be 0·001 at 20°K. Similar behaviour is shown by all elements though at different temperatures.

(2) At high temperatures the atomic heat converges to the value 3R in the case of most elements. In a few cases the melting point may be too high and hence at ordinary temperatures the atomic heat may appear too low (*e.g.*, for diamond), but at high temperatures the atomic heat of these substances increases rapidly towards the value 3R.

Nernst compared his data with Einstein's formula and found that though the law affords a qualitative explanation of the results yet it failed when treated quantitatively. But the deduction from Einstein's formula, namely, the point-to-point correspondence of the specific heat curves was found to hold good. This suggests that the true specific heat should be given by a formula of the general type.

$$mc_v = 3R\,\phi\left(\frac{\theta_m}{\theta}\right)$$

ϕ denoting a function different from Einstein's function. Nernst and Lindemann proposed the formula.

$$mc_v = \frac{3}{2}R\left[f\left(\frac{\theta_m}{\theta}\right) + f\left(\frac{\theta_m}{2\theta}\right)\right]$$

where f is Einstein's function.

The principal defect of Einstein's theory is that it ascribes to every element a certain frequency which is characteristic of that element, whereas it is very unlikely that in a solid mass the atoms will all continue to have the same frequency. A more rational

hypothesis is that the atoms vibrate with different frequencies according to the laws of statistical mechanics.

Debye's theory of the specific heat of solids.

An entirely new departure was made by Debye in 1912 in the case of monatomic solids.* Debye regards the solid as a continuous structure whose vibrations as a whole constitute the heat energy of the body and are governed by the laws of statistical mechanics. He shows that the supposition of Einstein that a solid has only a single characteristic frequency is not true but that a solid is characterised by a whole spectrum of characteristic frequencies. The frequencies can have all values between zero and a certain upper limit ν_m which is conditioned by the fact that matter is not infinitely fine-grained and which can be determined from the elastic constants of the body. It has been shown by Jeans and others that the number of any particular kind of stationary waves having frequencies lying between the limits ν and $\nu+d\nu$ that can exist in equilibrium in a certain volume of a continuous medium is $A\nu^2 d\nu$, where A is constant. Hence, the total number of heat waves that can exist in a volume containing a gram-atom of the solid is $\int A\nu^2 d\nu$. But the total number of degrees of freedom of N atoms in a gram-atom is 3N so that the maximum number of waves that can exist in a volume containing a gram-atom of the substance is 3N.

Thus we have

$$3N = \int_0^{\nu_m} A\nu^2 d\nu$$

$$\text{or } A = \frac{9N}{\nu_m^3}$$

According to Planck's law the amount of energy associated with each wave is

$$K\theta \frac{\frac{h\nu}{k\theta}}{e^{\frac{h\nu}{k\theta}}-1} - \frac{h\nu}{e^{\frac{h\nu}{k\theta}}-1}$$

Hence, the total energy E of a gram-atom is given by the formula

$$E = \frac{9N}{\nu_m^3} \int_0^{\nu_m} \frac{h\nu}{e^{\frac{h\nu}{k\theta}}-1} \nu^2 d\nu$$

* Debye, Ann. d. Phys. 39, 789 (1912),

$$= \frac{9Nh}{\nu_m^3} \int_0^{\nu_m} \frac{\nu^3 d\nu}{e^{\frac{h\nu}{k\theta}} - 1}$$

$$= \frac{9Nh}{\nu_m^3} \left(\frac{k\theta}{h}\right)^4 \int_0^{\frac{h\nu_m}{k\theta}} \frac{x^3 dx}{e^x - 1}$$

Since the exponent of e must be a pure number, $\frac{h\nu}{k}$ is of the dimensions of a temperature so that we may replace $\frac{h\nu}{k}$ by θ' and $\frac{h\nu_m}{k}$ by θ_m where θ_m is the temperature characteristic of the substance and is proportional to the highest frequency of the substance. Thus, we may also write

$$E = \frac{9R}{\theta_m^3} \int_0^{\theta_m} \frac{\theta'^3 d\theta'}{e^{\frac{\theta'}{\theta}} - 1}$$

Writing the energy of the gram-atom in the form

$$E = \frac{9R\theta^4}{\theta_m^3} \int_0^{x_m} \frac{x^3 dx}{e^x - 1}$$

where $x_m = \frac{\theta_m}{\theta} = \frac{h\nu_m}{k\theta}$

we have then,

$$mc_v = \frac{dE}{d\theta}$$

$$= \frac{9R}{\theta_m^3} \times 4\theta^3 \int_0^{x_m} \frac{x^3 dx}{e^x - 1} + \frac{9R\theta^4}{\theta_m^3} \cdot \frac{x_m^3}{e^{x_m} - 1} \times \left(-\frac{h\nu_m}{k\theta^2}\right)$$

$$= \frac{36R}{x_m^3} \int_0^{x_m} \frac{x^3 dx}{e^x - 1} - \frac{9Rx_m}{e^{x_m} - 1}$$

Debye gave the formula in the form

$$mc_v = 3R\left[\frac{12}{x_m^3}\int^{x_m}\frac{x^3dx}{e^x-1} - \frac{3x_m}{e^{x_m}-1}\right] = 3R\phi(x_m)$$

where ϕ is Debye's function.

Now, h is of the order 10^{-27},

and k is of the order 10^{-16},

and the maximum value of ν is generally found to be of the order 10^{12}. Hence at high temperatures $\frac{h\nu}{k\theta}$ is a very small quantity so that $e^x - 1 = x$ approximately.

Thus at high temperatures we may write

$$E = \frac{9R\theta^4}{\theta_m^3}\int^{x_m} x^2dx$$

$$= \frac{9R\theta^4}{\theta_m^3} \times \left(\frac{\theta_m}{\theta}\right)^3 \times \frac{1}{3} = 3R\theta$$

Hence at high temperatures

$$mc_v = \frac{dE}{d\theta} = 3R$$

At low temperatures $\frac{h\nu}{k\theta}$ = a large quantity so that we may regard the upper limit of the integration as infinity.

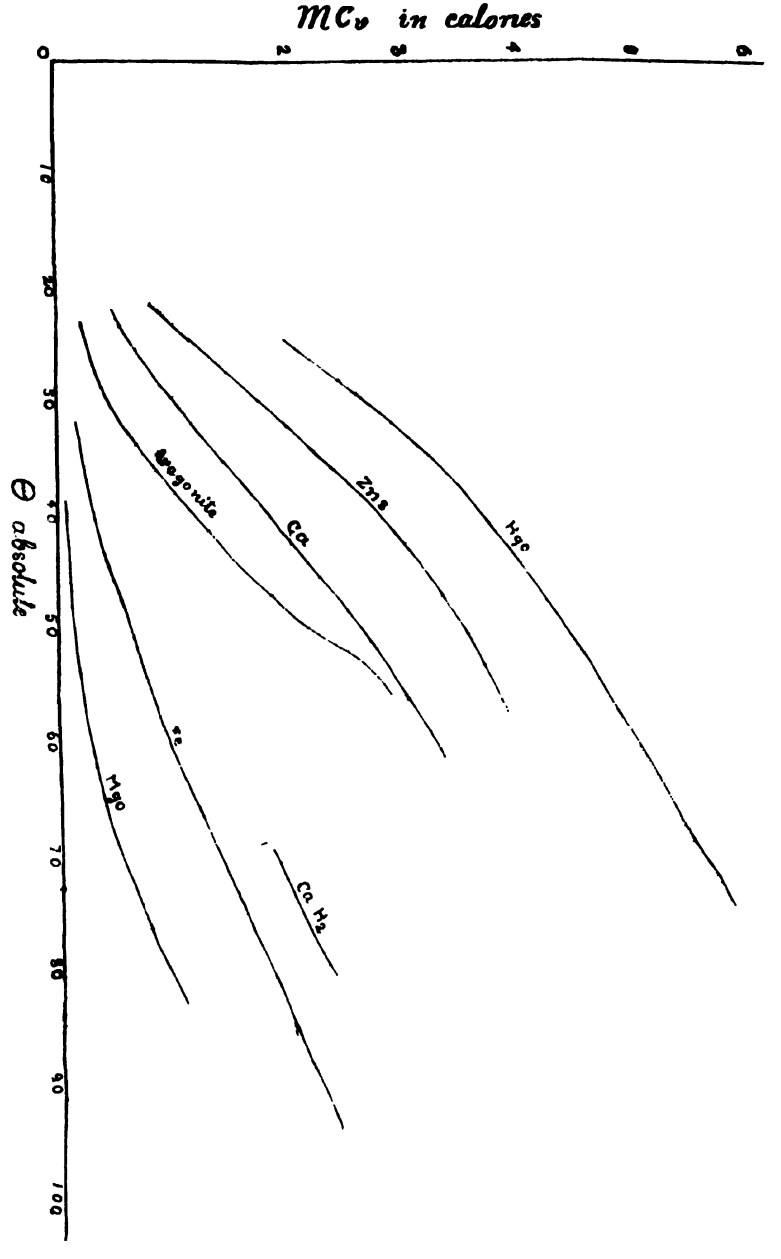
Thus, at low temperatures

$$E = \frac{9R\theta^4}{\theta_m^3}\int_0^{\infty}\frac{x^3dx}{e^x-1}$$

$$= \frac{9R\theta^4}{\theta_m^3} \cdot \frac{\pi^4}{15}$$

$$= \frac{3}{5}\frac{\pi^4 R\theta^4}{\theta_m^3} = a\theta^4,$$

where 'a' is a constant equalling $\frac{3}{5}\frac{\pi^4 R}{\theta_m^3}$. It can also be shown that


MC_v in calories
0
2
3
4
5
6
10
20
30
40
50
60
70
80
90
100
Θ absolute
Magnesite
Ca
Zns
Hgc
Mgo
Fe
Ca H₂

$E = \frac{9}{8\pi} (N\lambda^3 =) a\theta^4$, where a is the constant occuring in the formula for energy density of radiation within a Hohl raum.*

$$\text{Hence } mc_v = \frac{\partial E}{\partial \theta} = 4a\theta^3$$

According to this theory the specific heat of solids at low temperatures should vary as the third power of the absolute temperature. This law has been verified by Kammerlingh Onnes and Keesom and Eucken and Schwer. At the present time we can regard this law as the limiting law for all solids.

Debye calculated the value of the function ϕ for values of x_m from zero to infinity. He compared the results of his theoretical calculations with the specific heat data of Nernst and his collaborators and found an excellent agreement between the calculated and the experimental values.

Debye's formula for specific heat shows that mc_v is a universal function of $\frac{\theta}{\theta_m}$. Hence, if the value of θ_m be known for a substance the specific heat curve for that substance can be traced from the known value of the function ϕ throughout the whole range of temperatures. Hence, curves plotted with mc_v as ordinate and $\frac{\theta}{\theta_m}$ as abscissa should be identical for all solids. This is actually found to be true in the case of many metals. In other words it means that all solid substances have the same specific heat at temperatures which bear the same ratio to their characteristic temperatures.

Methods of calculating the characteristic temperature of solids from specific heat data.

The value of the characteristic temperature θ_m of a substance can be obtained in many ways. In the case of diamond, copper, silver, aluminium and lead the value of θ_m was calculated from the experimental data of Nernst and Lindemann.

* It may be interesting to note here that the total molecular energy $E = \frac{9}{8\pi} (3N\lambda_m^3) . a\theta^4$, where $a\theta^4$ is the density of black-body radiation,—in other words, the molecular energy is equivalent to the black body energy contained within a volume which is proportional to the volume occupied by 3N waves of length λ_m.

We know that the specific heat for solids at very low temperatures is given by the formula

$$c_0 = \frac{12}{5}\frac{\pi^4 R\theta^3}{\theta_m^3}$$

where c_0 = specific heat at a very low temperature θ, and that at very high temperatures it is given by the formula

$$c_\infty = 3R$$

where c_∞ = specific heat at very high temperatures.

Combining these two results we have

$$\frac{c_0}{c_\infty} = \frac{4\pi^4}{5}\left(\frac{\theta}{\theta_m}\right)^3$$

$$\text{or } \theta_m = \left(77{\cdot}94\,\frac{c_\infty}{c_0}\right)^{\frac{1}{3}}\theta.$$

This affords one method of calculating θ_m from the known values of the specific heat of a substance at a very high temperature and also at a very low temperature.

The value of θ_m may also be obtained by trial. In the following table, column (4) gives the value of θ_m obtained according to the first of these two methods and column (5) gives the value of θ_m obtained by trial from the specific heat data.

Table IV.

Element.	θ	$\frac{c_0}{c_\infty}$	θ_m	θ_m
Diamond.	20·5	0·104	1860	1830
Aluminium.	32·4	0·042	398	396
Copper.	23·5	0·037	302	309

Debye's Method of calculating ν_m.

Debye calculated the value of θ_m from the elastic constants of the solid, for, according to Debye, the vibrations in a solid giving rise to its thermal energy are identical with the elastic vibrations of the solid body. Elastic waves are of two types, namely, compressional waves and distortional waves. Also, each distortional wave has two com-

ponents. It is known from statistical mechanics that the number of waves of any kind having frequencies lying between the limits ν and $\nu+d\nu$ which can exist in a volume V is

$$\frac{4\pi V\nu^2 d\nu}{a^3}$$

where 'a' is the velocity of the waves. Hence, the total number of compressional and distortional waves in a volume V of the solid is

$$4\pi V\left(\frac{1}{a_1^3}+\frac{2}{a_2^3}\right)\nu^2 d\nu$$

where a_1 = velocity of the compressional wave,
and a_2 = velocity of the distortional wave.

If the volume V of the solid contains a gram-atom and if N be the Avagadro number, the maximum number of vibrations that can exist in the volume V is 3N. Hence,

$$3N=4\pi V\int_0^{\nu_m}\left(\frac{1}{a_1^3}+\frac{2}{a_2^3}\right)\nu\, d\nu$$

$$=\frac{4\pi V}{3}\nu_m^3\left(\frac{1}{a_1^3}+\frac{2}{a_2^3}\right)$$

so that $\nu_m^3=\dfrac{9N}{4\pi V\left(\dfrac{1}{a_1^3}+\dfrac{2}{a_2^3}\right)}$

Also, $a_1=\sqrt{\dfrac{k+\frac{4}{3}\eta}{\rho}}$

and $a_2=\sqrt{\dfrac{\eta}{\rho}}$

where k = modulus of bulk elasticity of the solid,
η = rigidity,
ρ = density, $\rho=\dfrac{M}{v}$, M = Atomic weight.

Hence, since all the other constants are known ν_m (and hence θ_m) can be calculated. In the following table, column II shows the values of θ_m obtained by Debye from elastic constants and column III indicates the values of θ_m obtained from specific heat data. The agreement is quite good except in the case of lead for which the elastic constants are not accurately known.

TABLE V.

Element.		θ_m	
Aluminium	...	399	396
Copper	...	329	309
Silver	...	212	215
Lead	...	95	72

Another method of calculating ν_m is due to Lindemann. He showed that the characteristic frequency ν_m is connected with the melting point T. by the following formula.

Lindemann's method of calculating ν_m.

$$\nu_m = K \sqrt{\frac{T_s}{MV^3}}$$

where K=a constant
M=atomic weight of the solid
V=gram-atomic volume of the solid.

Lindemann supposes that at the stage of melting the amplitude of vibration of an atom of the solid is comparable to the mean distance between two contiguous atoms.* Suppose that at a particular instant the displacement of the atom from its mean position is x.

Then, the equation of displacement is

$$m\ddot{x} + \mu x = o,$$

where μ=a constant.

m=atomic weight of the solid.

The solution of this equation is

$$x = a \cos (nt + \alpha) \text{ where } n^2 = \frac{\mu}{m}.$$

Hence $\nu = \frac{1}{2\pi} \sqrt{\frac{\mu}{m}}$

The energy of the atom is

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}m\dot{z}^2 + \tfrac{1}{2}\mu x^2 + \tfrac{1}{2}\mu y^2 + \tfrac{1}{2}\mu z^2.$$

* It may be remembered here that X-ray Diffraction Patterns from crystals become blurred at the melting point temperatures of the crystals and this supports the supposition of Lindemann.

Also, it can be proved that $m\dot{x}^2 = \mu x^2 = \frac{\mu a^2}{2}$ and so on.

Hence, the total energy E of the vibrating atom is $\frac{3}{2}\mu a^2$, where a=amplitude of vibration of the atom.

But since the vibrating atom has three degrees of freedom

$$E = 3k\theta.$$

Thus, from these two values of E we have

$$\mu = \frac{2k\theta}{a^2}$$

Again, a^3 = volume occupied by one atom

$$= \frac{V}{N}, \text{ where N = Avagadro number.}$$

Hence, $\mu = 2k\theta \cdot \frac{1}{\left(\frac{V}{N}\right)^{\frac{2}{3}}}$

But $= \frac{1}{2\pi}\sqrt{\frac{\mu}{m}}$

Hence, substituting the value of M and remembering that $\theta = T$, and $M = Nm$ we have

$$\nu = K\sqrt{\frac{T}{MV^{\frac{2}{3}}}}$$

woere K is a constant.

According to Lindemann the frequency thus obtained is the characteristic frequency of the solid so that we may write

$$\nu_m = K\sqrt{\frac{Ts}{MV^{\frac{2}{3}}}}.$$

Besides the methods mentioned above there are other methods of finding the characteristic frequency of a solid. Thus, it is found that the frequency of the largest 'Rest-strahlen' given by a solid coincides with its characteristic frequency ν_m. Hence, a determination of the frequencies of the 'Rest-strahlen' affords a means of determining ν_m.

The following tables give some of the results of Nernst and his co-workers.

TABLE VI.

Diamond (Melting point—3600°)

θ	mc_p	$m\Delta c_p$	mc_v	$\frac{c_v}{c^\infty}$
30	0·00	.....	0·00	0·000
42	0·00	.....	0·00	0·000
88	0·03	.....	0·03	0·005
92	0·03	.....	0·03	0·005
205	0·62	.....	0·62	0·104
209	0·66	.....	0·66	0·111
220	0·72	.....	0·72	0·121
222	0·76	.....	0·76	0·128
232	0·86	.....	0·86	0·145
243	0·95	.....	0·95	0·160
262	1·14	.....	1·14	0·191
284	1·35	.....	1·35	0·227
306	1·58	.....	1·58	0·266
331	1·84	0·01	1·83	0·308
358	2·12	0·01	2·11	0·354
413	2·66	0·07	2·64	0·444
1169	5·48	0·21	5·26	0·880

mc_∞ = limiting atomic heat for solids = 5·96.

θ = temperature of solid on the absolute scale.

mc = atomic heat at constant volume.

$m\Delta c = m(c - c)$.

TABLE VII.

Aluminium.

θ	Mc_p	$M\Delta c_p$	Mc_v	$\frac{c_v}{c_\infty}$
32·4	0·25	...	0·25	0·042
35·1	0·33	...	0·33	0·055
83·0	2·41	0·01	2·40	0·403
86·0	2·52	0·01	2·51	0·421
88·3	2·62	0·01	2·61	0·438
137	3·97	0·04	3·93	0·660
235	5·32	0·15	5·17	0·868
331	5·82	0·25	5·57	0·935
433	6·10	0·35	5·75	0·966
555	6·48	0·52	5·98	1·005

TABLE VIII.

Copper.

θ	Mc_p	$M\Delta c_p$	Mc_v	$\frac{c_v}{c_\infty}$
23·5	0·22	...	0·22	0·037
27·7	0·32	...	0·32	0·054
33·4	0·54	...	0·54	0·092
87·0	3·33	0·01	3·32	0·558
88·0	3·38	0·01	3·37	0·565
137	4·57	0·04	4·53	0·761
234	5·59	0·09	5·50	0·924
290	5·79	0·13	5·66	0·951
323	5·90	0·15	5·75	0·966
450	6·09	0·22	5·87	0·985

TABLE IX.

Silver.

θ	Mc_p	$M\Delta c_p$	Mc_v	$\frac{c_v}{c_\infty}$
35·0	1·58	...	1·58	0·266
39·1	1·90	...	1·90	0·319
42·9	2·26	...	2·26	0·380
45·5	2·47	0·01	2·46	0·413
51·4	2·81	0·01	2·80	0·470
53·8	2·90	0·01	2·89	0·502
77·0	4·07	0·03	4·04	0·678
100	4·86	0·06	4·80	0·806
200	5·78	0·17	5·61	0·944
273	6·00	0·25	5·75	0·965
331	6·01	0·30	5·71	0·960
535	6·46	0·56	5·90	0·990
589	6·64	0·65	5·99	1·006

TABLE X.

Lead.

θ	Mc_p	ΔMc_p	Mc_v	$\frac{c_v}{c_\infty}$
23·0	2·96	0·01	2·95	0·495
28·3	3·92	0·01	3·91	0·656
36·8	4·40	0·02	4·38	0·735
38·1	4·45	0·02	4·43	0·744
85·5	5·65	0·08	5·57	0·935
90·2	5·71	0·09	5·62	0·945
200	6·13	0·22	5·91	0·993
273	6·31	0·34	5·97	1·002
290	6·33	0·35	5·98	1·004
332	6·41	0·41	6·00	1·007
409	6·61	0·54	6·07	1·02

TABLE XI.

Fe $\frac{h\nu_0}{k} = 453$ (Debye).

θ	Mc$_v$ (observed).	Mc$_v$ (calculated).
32·0	0·15	0·16
33·1	0·18	0·18
35·2	0·24	0·21
38·1	0·29	0·27
42·0	0·33	0·36
46·9	0·52	0·51
66·8	1·21	1·23
68·9	1·48	1·32
85·3	1·94	1·99
87·1	2·04	2·06
88·9	2·22	2·14
94·6	2·36	2·36

TABLE XII.

Ca

θ	Mc$_v$ (observed).	Mc$_v$ (calculated).
22·3	0·51	0·46
24·0	0·63	0·56
25·9	0·74	0·69
29·2	0·89	0·89
32·6	1·12	1·15
35·8	1·42	1·42
38·8	1·57	1·67
46·3	2·13	2·27
49·3	2·24	2·50
52·0	2·59	2·70
56·5	2·82	2·99
59·7	3·06	3·20
62·0	3·20	3·31

TABLE XIII.

CaH_2 $\frac{h\nu_m}{k} = 391$ (Debye).

θ	Mc_v (observed).	Mc_v (calculated).
69·9	1·79	1·82
71 2	1·86	1·89
72·5	1·90	1·93
79·0	2·25	2·24
80·1	2·37	2·30
80·9	2·37	2·34

TABLE XIV.

Zinc blende $\frac{h\nu_m}{k} = 175$ (Debye).

θ	Mc_v (observed).	Mc_v (calculated).
21·6	0·74	0·81
25·0	1·13	1·14
28·0	1·51	1·47
30·6	1·73	1·74
32·7	2·02	1·98
34·9	2·29	2·20
36·8	2·43	2·40
38·7	2·70	2·58
44·2	3·01	3·04
48·0	3·10	3·31
51·1	3·58	3·53
53·9	3·64	3·69
56·9	3·75	3·85
58·6	4·00	3·92

TABLE XV.

Mgo $\frac{h\nu_m}{k}$ =768 (Debye)

$\frac{h\nu_m}{k}$ =552 (Einstein).

θ	Mc_v (observed).	Mc_v (calculated).
39·2	0·038	0·062
42·6	0·095	0·090
52·7	0·151	0·151
56·5	0·202	0·218
(64·4	0·394	0·336)
65·8	0·383	0·391
81·2	0·982	0·988
83·3	1·019	1·059

The two adjoining curves are the temperature-specific heat curves obtained from Nernst's experimental data. They clearly indicate how the specific heat of solids varies at low temperature. The curve for diamond is very interesting. Diamond is one of those elements which do not obey Dulong and Petit's law at ordinary temperatures. From the curve it is evident that if the specific heat of diamond be determined at a sufficiently high temperature the atomic heat of diamond will be equal to the Dulong and Petit's constant 6·4.

The following two tables (tables XVI and XVII) indicate the variation of the atomic heat with the variation of $\frac{\theta_m}{\theta}$. The first of them gives results according to Einstein's formula and the second according to Debye's formula.

TABLE XVI.

MC, according to Einstein.*

$$\frac{h\nu_m}{k\theta} = \frac{\theta_m}{\theta} \text{ from 0 to 14.}$$

$\frac{h\nu_m}{k\theta}$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0	5·965	5·947	5·935	5·911	5·878	5·833	5·780	5·718	5·648	5 568	5·483
1	5·483	5·401	5·279	5·184	5·071	4·954	4·832	4·706	4·578	4·446	4·312
2	4·312	4 176	4·039	3·902	3·764	3·626	3·489	3·353	3·218	3·086	2·954
3	2·954	2·827	2·701	2·578	2·458	2·342	2·229	2·119	2 013	1·910	1·811
4	1·811	1·715	1·623	1·536	1·451	1·370	1·292	1·218	1·148	1·081	1·017
5	1·017	0·956	0·898	0·843	0·791	0·743	0 696	0·652	0·610	0·571	0·535
6	0·535	0·449	0·466	0·436	0·407	0·379	0·354	0·330	0·307	0 286	0·266
7	0·266	0·248	0·231	0·215	0·200	0·185	0·172	0·160	0·149	0·138	0·128
8	0·128	0·119	0·110	0·102	0·0945	0·0884	0·0811	0·0752	0·0695	0·0650	0·0600
9	0·0600	0·0554	0·0509	0·0468	0·0435	0·0400	0·0372	0·0340	0·0310	0·0286	0·0266
10	0·0266	0·0247	0·0231	0·0212	0·0196	0·0180	0·0167	0·0152	0·0142	0 0129	0·0119
11	0·0119	0·0110	0·0102	0·0093	0·0085	0 0078	0·0073	0·0067	0·0062	0·0057	0·0052
12	0·0052	0 0048	0·0045	0·0041	0·0038	0·0035	0·0032	0·0029	0·0027	0·0024	0·0022
13	0·0022	0·0020	0·0019	0·0017	0·0016	·0015	0·0014	0·0013	0·0012	0·0011	0·0010

* This table is taken from Nernst's "Theoretischen Une Experimentellen Grundlagen Des Neuen Warmesatzes."

TABLE XVII.

MC, according to Debye.*

$$\frac{h\nu_m}{k\theta} = \frac{\theta_m}{\theta} \text{ from 0 to 30}$$

$\frac{h\nu_m}{k\theta}$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0	5·955	5·95	5·94	5·93	5·91	5 88	5·85	5·81	5·77	5·73	5·67
1	5·670	5·61	5·54	5·48	5·41	5·34	5·26	5·18	5·08	5·00	4·91
2	4·914	4·82	4·73	4·64	4·54	$4·43_7$	4·34	4·24	4·14	4·04	4·94
3	$3·94_1$	3·84	3·74	3·65	3 55	3·45	3·36	3·27	3·18	3 09	3·00
4	$2·99_6$	2·91	2·83	2·75	2 67	2·59	2·51	2·43	2·35	2·27	2·20
5	$2·19_8$	2·13	2·06	2 00	1·94	1·88	1·82	1·76	1·70	1·64	1·58
6	$1·58_1$	1·53	1·48	1·43	1·38	1·34	1·30	1·26	1·22	1·18	1·14
7	1·137	1·102	1·067	1·033	1·000	0·968	0·937	0·907	0·879	0·853	0·830
8	0·830	0 810	0·790	0·770	0·750	0·731	0·712	0·693	0 674	0·655	0·636
9	0·636	0·617	0·598	0·579	0·560	0·541	0·523	0.505	0·487	0·469	0·451
10	0·451	0·435	0·421	0·409	0·398	0·388	0·378	0·368	0·359	0·351	0 343
11	0·343	0·335	0·327	0·319	0·311	0·303	0·295	0·287	0·280	0·273	0·267
12	0·267	0·261	0·255	0·249	0·243	0·237	0·231	0·225	0·220	0·215	0·210
13	0·210	0·205	0·200	0·196	0·192	0·188	0 184	0·180	0·176	0·172	0·168
14	0·168	0·164	0·161	0·158	0·155	0·152	0·149	0·146	0·143	0·140	0·137
15	0·137	0·134	0·131	0·128	0·125	0·123	0·121	0·119	0·117	0·115	0·113

* This table is taken from Nernst's "Theoretischen Une Experimentellen Grundlagen Des Neuen Warmesatzes."

MC, according to Debye.

$\frac{h\nu_m}{k\theta}$		$\frac{h\nu_m}{k\theta}$		$\frac{h\nu_m}{k\theta}$	
16	0·113	21	0·0499	26	0·0262
17	0·094	22	0·0433	27	0·0234
18	0·079	23	0·0380	28	0·0211
19	0·067	24	0·0335	29	0·0189
20	0·057$_9$	25	0·0296	30	0·0172

BOTANY

BAL.

Pseudoperonospora cubensis on *Trychosanthes dioica.*

BAL. CHOUDHURY.

B C

A

D

Cercospora personata on *Arachis hypogaea*

BAL. BANERJEE.

COMMENTATIONES MYCOLOGICÆ

8.—*Pseudoperonospora cubensis* (B. AND C.), ROSTEN, ON *Trichosanthes dioica* ROXB.

BY

S. N. BAL.

Trichosanthes dioica (Beng. Patol) is one of the cucurbitaceæ, which is largely cultivated in Bengal and the fruits of which are greatly relished as a vegetable.

The fungus, to be described, has a peculiar historical record. In 1868 it was first described from Cuba as growing on a wild plant and it was not recorded from any other country till 1889, when it was described from New Jersey (U. S. A.), as occurring on the leaves of greenhouse cucumbers; it also had been collected in Japan in 1888. Later on, it became a very common parasite on the cucurbits of the United States. It was reported from Europe only as recently as in 1899, after which it has been reported from almost all countries—from Brazil in 1900, Russia and Japan in 1902, Austria, Italy and East Africa in 1903, China in 1908 and Australia and India in 1910. It is not known when it reached India. Butler says, "An unidentified Peronospora was recorded in Ceylon in 1901 and as the present one is the only member of the family known on this host, it is quite possible that the disease has reached us from the south." It, however, has not previously been reported from Bengal.

During the middle of the last month, after a shower of rain, the writer noticed that the leaves of some of the Patol plants in a kitchen garden were attacked by some fungus, as they showed signs of decay. He searched for similar fungal attacks on the other cucurbits, namely Pepo and Bottlegourds, growing quite close in the same garden, but they were not found to be attacked.

The attack, on *Trichosanthes dioica* is confined to the leaves. Small pale yellow-coloured patches were first observed, which later became brownish yellow and at a still later stage the spots united to form larger patches. At last the attacked portions of the leaf dried up and the whole leaf became yellow, withered and fell off. The older leaves appear to be attacked first, the spread of the disease being

rather rapid and within 10 days the plants have left only a tuft of leaves at the tip of the creeping stem and appear so sickly that it is not probable that fruits will be obtained from them. It is probably due to the prevalent favourable conditions, namely, little sunshine, cloudy morning and comparatively low temperatures, which favoured the consequent development of the fungus. The fungus, as reported from Pusa, is stated to have not done very serious damage to the cucurbits there, the reason probably being that the growth of the fungus is not particularly rapid there.

On examination of sections, cut through the diseased portion of the leaves, it is found that the mycelium consists of non-septate hyphæ which are branched and irregular, penetrating the mesophyll cells. The sporangiophores arise from these hyphæ emerging mainly from the stomata of the lower surface of the leaves. These sporangiophores vary greatly in length (180 to 320 μ), the breadth below the first branching being 5 to 7 μ. The branching of the sporangiophores is roughly dichotomous. The ultimate branches taper to a rather blunt apex, on which is borne a single sporangium.

The sporangia are quite hyaline and ovoid or ellipsoidal in shape, having a distinct surrounding wall. The mature sporangia are of the same shape, but they are rather of a greyish colour, having a surrounding wall and a small hyaline papilla at the open end and measure 18 to 25 × 12 to 18 μ. They fall off when ripe and germinate easily in water, usually giving out a zoospore or sometimes a germ tube.

The zoospores are kidney-shaped and biciliate and after coming to rest they measure 9 to 11 μ across. Infection may occur from both the surfaces of the leaf, germ-tubes or zoospores being able to penetrate the epidermal cells or the stomata. No Oospore has been found.

Butler says that the spread of the disease is greatly influenced by the humidity and perhaps also by the temperature. Low temperatures probably favour germination. Cloudy or foggy, cool days with morning dew and little sun are supposed to be the most dangerous to the crop, infected with the fungus. In the present case these conditions were all more or less prevalent.

It is reported from other countries that cucumber and melon plants are readily killed. As the disease has already spread in Bengal where the cucurbits are grown almost all the year round and as the fungus is liable to attack any of these cucurbits, timely precautions should be taken.

The remedial measure, adopted in the United States, is spraying with Bordeaux mixture. Butler thinks this mixture to be rather hurtful to some of the cucurbits and advises the use of a mixture of 2 lbs. of copper sulphate and 2 lbs. of lime to 50 gallons of water. He also says spraying should commence about the time the disease first appears and be continued at intervals of 14 days or so.

Explanation of the Plate.

1. A leaf of *Trichosanthes dioica*, infected with the fungus, Natural size.
2. A branching sporangiophore bearing two immature sporangia, × 440.
3. Mature sporangia, × 440.
4. A zoospore, × 570.

Botanical Laboratory,
University College of Science.
Calcutta, the 3rd April 1920.

Commentationes Mycologicæ

9.—*Cercospora Personata* (B. & C.), Ellis, on leaves of *Arachis hypogœa*, Linn.

BY

S. N. Bal.

Groundnut, as known, is a native of Brazil. It has not been cultivated in India earlier than about 1850. Dr. Dymock thinks that the plant reached India through China. It is generally cultivated throughout India, but chiefly in South India and Bombay. It is best grown on good dry and sandy soil.

It is of comparatively recent introduction into Bengal, and experiments were first made with it in the almost barren highlands in Bankura, Birbhum and Midnapore, where it is doing quite well. Recently experiments have been successfully made with the crop in parts of Nadia, Murshidabad and Jessore Districts.

When collecting fungi during the month of November, last year, some sickly plants of Groundnut were observed in a garden at Ballyganj. On enquiry the writer learnt that some of the bazar nuts were planted there for experimental purposes.

The few leaves that were on the plants were almost all spotted and a heap of withered and spotted leaves was found on the ground.

The lower leaves are attacked first. A number of dark spots appear on the surface of the leaves, surrounded by bright yellow ring. The spots on a single leaf vary from 3 to 15. The shape of the spots is generally round but sometimes two or more are confluent making the shape irregular. The size of the single spots varies from 2 mm. to 8 mm. in diameter. All parts of the leaves including midribs and veins have been found to be attacked. These spotted leaves soon fall off, leaving the plant almost leafless. At a later stage of the infection the remaining green portion of the attacked leaves become yellowish.

On examination of sections, cut through the diseased portion of the leaves, it is found that the mycelium penetrates the leaf tissue and is composed of branched hyphæ running between the leaf cells

and at some places sending out haustoria. The upper cells of the hypha are protruded, and by their pressure rupture the epidermis and a mass of conidiophores appear on the surface.

The conidiophores are short (up to 25 μ in length and 5 to 7 μ in breadth). They are unbranched and nonseptate. They have each at their apex a slight bend showing the point of attachment of a spore. The spores are clavate, pale brown in colour, transversely septate forming 3 to 5 divisions. They measure 30 to 50 $\times$ 5 to 6 μ.

The germination of the spores was easily effected in water, when after 24 hours they gave out a germ tube from near one end of the spore. Inoculation experiments were successfully carried out on healthy groundnut leaves in a moist chamber by spraying them with the water in which spores had been germinated,—new spots appeared after 6 days.

The disease caused by the fungus is a serious disease of the groundnut and is commonly known as "Tikka disease." It caused a serious damage to the groundnut cultivation in the Bombay Presidency and it has been thoroughly studied by Dr. Butler, who published the results of his investigations in "The Agricultural Journal of India, Vol. IX, 1914," under the title of "Tikka disease and the introduction of exotic Groundnuts in the Bombay Presidency." The disease has not been previously reported from Bengal.

Butler in his book, "Fungi and Disease in Plants," says that spraying has not given good results in India and he suggests seed disinfection,—4 hours in Formalin in a strength of 1 lb. to 40 gallons of water or half hour in $\frac{1}{2}$ % copper sulphate solution. In Bombay, however, the groundnut cultivation, which almost died out, has been successfully revived by the introduction of exotic varieties. The decline in cultivation was however due to the ravages of the "Tikka disease."

Wolf also studied the disease as occurring in the United States of America and published the results of his investigations under the title of "Leafspot and some fruit rots of Peanuts," in 1914 (Alabama Agricultural Experiment Station Bulletin, No. 180, 1914). He further studied the question in 1916 and published another account in "The Journal of Agricultural Research, Washington, Vol. V, Jan. to March, 1916," under the title of "Further studies of Peanut leafspot." He concludes that neither rotation nor disinfection of seeds prevent the disease.

The spores of the fungus are very resistant; they are wind-borne and also, as proved by Wolf, are transmitted by insect inhabiting the Groundnut fields. It may therefore be expected that the disease is readily spread by the insects inhabiting fields where it is already prevalent.

EXPLANATION OF THE PLATE.

A. An infected leaf of *Arachis hypogæa*, Linn.,—Natural size.

B. A portion of a transverse section cut through an infested portion of a leaf showing the conidiophores and haustoria, × 440.

C. A conidiophore of the fungus bearing a conidium, × 570.

D. A few spores of the fungus, × 440.

BOTANICAL LABORATORY,
UNIVERSITY COLLEGE OF SCIENCE,
Calcutta, the 3rd April, 1920.

Commentationes Mycologicæ

10.—*Rhinocladium corticolum*, Mass., on the bark of *Mangifera indica*, Linn.

BY

S. N. Bal & K. G. Banerjee.

Towards the end of the month of January of the present year the mango trees, standing on the College grounds, were noticed to be infested with some black patches here and there on the bark of both the old stumps and the smaller branches. On examination of these diseased portions of the bark it was found to be attacked by a fungus.

The patches are 2·5 to 12 cm. long, and on the younger branches they sometimes encircle the whole of the portion thus attacked, generally at the base of the tertiary branches. The infection is superficial and practically does no harm to the trees.

The fungus forms a black layer over the infested portion of the bark of the affected branch, the spots consisting of mycelium, at places forming tufts, which may be close together or a little apart. These acervuli are sometimes orbicular, but more often irregular in shape. The hyphæ are dark coloured and septate. The sterile hyphæ are quite abundant as compared with the fertile ones and they are often twisted together in bundles, which are generally erect or slightly curved. The fertile hyphæ are rather scarce, less dark in colour and they bear the conidiophores. The conidia are borne on short structures, resembling sterigmata, at the apices of the fertile hyphæ; they are globose and are of a much lighter colour than the mycelium. They measure 12 to 16 μ in diameter. The episporium is thick and very densely and minutely tuberculate. The colour of the hyphæ becomes gradually paler towards the apices.

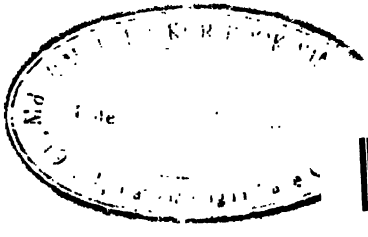
The fungus has been described by Massee in 1901 (Kew Bulletin, 1901, p. 153); the specimen is mentioned to have been obtained from the Bombay Presidency.

EXPLANATION OF THE PLATE.

1. A portion of the branch of a mango tree attacked by *Rhinocladium corticolum*, Mass.,—Natural size.
2. Serile hyphæ, × 100.
3. A portion of sterile hyphæ, × 440.
4. A portion of the fertile hyphæ bearing the conidiophores and conidia, × 440.
5. Conidiophores bearing the conidia, × 440.

BOTANICAL LABORATORY,
UNIVERSITY COLLEGE OF SCIENCE,
Calcutta, the 3rd April, 1920.

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